3/06/2016
ENPC - Operations Research and Transport

You have 2 hours for the exam. Exercises are independent. Computer, phones, tablets and every connected objects are forbidden. Every note is allowed.

Exercice 1 (3pts). *Consider a game where rewards (to be maximized) are given by the following table where actions of player* 1 *correspond to the lines, actions of player* 2 *to the columns, rewards being given in the order of player.*

	a	b	c
a	(1,2)	(4,3)	(4,4)
b	(2,6)	(5,5)	(2,6)
c	(3,2)	(2,1)	(1,1)

1. *Find the Nash equilibrium(s)*
2. *Find the social optimum(s)*
3. *Find the Pareto optimum(s)*

Exercice 2 (5pts). *Consider the following non-oriented weighted graph.*

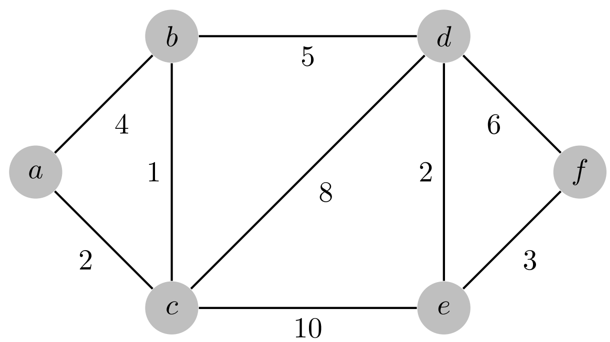

Figure 1: non-oriented graph

1. *Why can Dijkstra's algorithm be used to find a shortest path on this graph (Figure 1) ?*
2. *Use Dijkstra's algorithm to find the shortest path between node a and node f. The results can be presented in a table of the labels where each column corresponds to a node of the graph, and each line to an iteration of the Dijkstra algorithm. Give the shortest path and its cost.*
3. *Can we find a topological order for this graph (Figure 1) ? for the next one (Figure 2) ?*
4. *Find the shortest a-f-path for the graph in Figure 2.*

Exercice 3 (6pts). *We consider the problem*

$$\min_{x\in[-1,1]^2} \quad J(x) := (x_1 - 2)^2 + (x_2 + 1)^2.$$

We want to solve this problem through Frank-Wolfe algorithm (a.k.a Convex Combination Method).

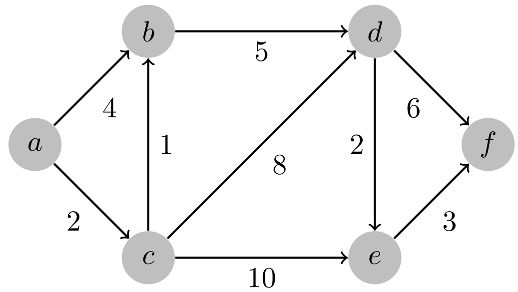

Figure 2: oriented graph

1. *Will the Frank-Wolfe algorithm converge ?*

2. *Compute the gradient of J.*

3. *Assume that $x^{(0)} = (0, 0)$. Write and solve the linear problem that is part of the first iteration of the Frank-Wolfe algorithm.*

4. *Write and solve the linear search problem that is part of the first iteration of the Frank-Wolfe algorithm. Find the new point $x^{(1)}$.*

5. *Write the linear problem part of the second iteration of the Frank-Wolfe algorithm.*

Exercice 4 (6pts)**.** *The price of anarchy of a class $\mathcal{C}$ of cost functions is the highest price of anarchy obtained by choosing any (finite) graph, with any rate (i.e. input/output flow) and any cost function in the class.*

We will assume that $\mathcal{C}$ contains the constant functions. Moreover we assume that $\mathcal{C}$ contains only non-decreasing functions.

1. *What is the price of anarchy if $\mathcal{C}$ is the class of non-decreasing affine functions?*

2. *What is the price of anarchy if $\mathcal{C}$ is the class of non-decreasing polynomial functions?*

A Pigou network is a network with two nodes: one origin o, one destination d, and two arcs linking o to d, and a flow rate between o and d of $r > 0$. The cost function of the first arc is $c \in \mathcal{C}$, and the cost function of the second arc is constant equal to $c(r)$.

3. *What is the user equilibrium, social optimum and price of anarchy of a Pigou network with given rate $r > 0$? (The result is not a closed formula but contains a max or a min).*

4. *Deduce a lower bound of the price of anarchy of the class $\mathcal{C}$.*

5. *Show that this lower bound is exact for the class of affine non-decreasing functions.*

Exercice 5 (bonus)**.** *Give* 3 *optimization problems tackled by Air France-KLM Operations Research group.*