3/06/2016
ENPC - Operations Research and Transport

You have 2 hours for the exam. Exercises are independent. Computer, phones, tablets and every connected objects are forbidden. Every note is allowed.

Exercice 1 (3pts). *Consider a game where rewards (to be maximized) are given by the following table where actions of player* 1 *correspond to the lines, actions of player* 2 *to the columns, rewards being given in the order of player.*

	a	b	c
a	(1,2)	(4,3)	(4,4)
b	(2,6)	(5,5)	(2,6)
c	(3,2)	(2,1)	(1,1)

1. *Find the Nash equilibrium(s)*
2. *Find the social optimum(s)*
3. *Find the Pareto optimum(s)*

Solution.

1. *(1pt) NE :* (a, c) *and* (c, a)
2. *(1pt) SO :* (b, b)
3. *(1pt) Pareto :* (b, a), (b, b), (b, c).

Exercice 2 (5pts). *Consider the following non-oriented weighted graph.*

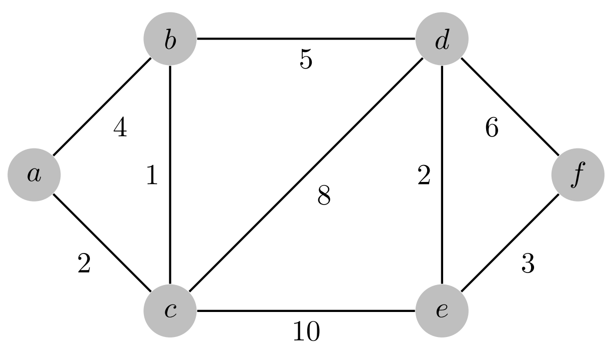

Figure 1: non-oriented graph

1. *Why can Dijkstra's algorithm be used to find a shortest path on this graph (Figure 1) ?*
2. *Use Dijkstra's algorithm to find the shortest path between node* a *and node* f. *The results can be presented in a table of the labels where each column corresponds to a node of the graph, and each line to an iteration of the Dijkstra algorithm. Give the shortest path and its cost.*
3. *Can we find a topological order for this graph (Figure 1) ? for the next one (Figure 2) ?*

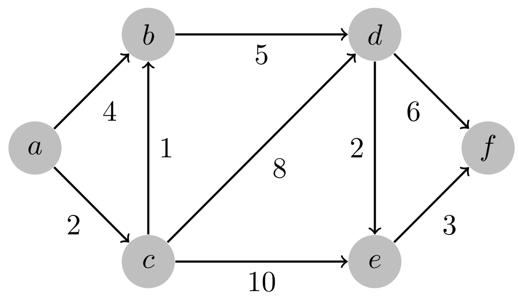

Figure 2: oriented graph

4. *Find the shortest a-f-path for the graph in Figure 2.*

Solution. *1. (0.5pt) We can consider that each arc is two opposite directed arcs with positive costs. Hence, Djikstra algorithm applies.*

2. *We have (2 pt)*

a	b	c	d	e	f
(0)	∞	∞	∞	∞	∞
0	(4)	(2)	∞	∞	∞
0	(3)	2	(10)	(12)	∞
0	3	2	(8)	(12)	∞
0	3	2	8	(10)	(14)
0	3	2	8	10	(13)
0	3	2	8	10	13

(0.5pt) Thus the shortest path is $a - c - b - d - e - f$ for a cost of 13.

3. *(1pt) No for Figure 1 graph is non-oriented. Yes for figure 2 : a-c-b-d-e-f.*

4. *(1pt) Graph 2 is a subset of graph 1 dedoubled, and the shortest path in graph 1 is admissible in graph 2, hence it is still the shortest path.*

Exercice 3 (6pts)**.** *We consider the problem*

$$\min_{x\in[-1,1]^2} \quad J(x) := (x_1 - 2)^2 + (x_2 + 1)^2.$$

We want to solve this problem through Frank-Wolfe algorithm (a.k.a Convex Combination Method).

1. *Will the Frank-Wolfe algorithm converge ?*

2. *Compute the gradient of J.*

3. *Assume that $x^{(0)} = (0, 0)$. Write and solve the linear problem that is part of the first iteration of the Frank-Wolfe algorithm.*

4. *Write and solve the linear search problem that is part of the first iteration of the Frank-Wolfe algorithm. Find the new point $x^{(1)}$.*

5. *Write the linear problem part of the second iteration of the Frank-Wolfe algorithm.*

Solution. 1. *(0.5pt) Objective function is (strongly) convex, constraints are polyhedral.*

2. *(1pt)* $\nabla J(x) = (2(x_1 - 2), 2(x_2 + 1))^\top$

3. *(1pt) The linear problem is given by*

$$\min_{y \in [-1,1]^2} -4y_1 + 2y_2$$

with optimal solution $(1, -1)$.

4. *(1pt) The linear search problem is*

$$\min_{t \in [0,1]} (t-2)^2 + (-t+1)^2.$$

(1pt) The unconstrained problem has an optimal solution of $t = 3/2$, *hence the optimal* t *is* $t = 1$, *and we have (0.5pts)* $x^{(1)} = (1, -1)$.

5. *(1pt) The new linear problem is given by*

$$\min_{y \in [-1,1]^2} -2y_1.$$

Exercice 4 (6pts). *The price of anarchy of a class* $\mathcal{C}$ *of cost functions is the highest price of anarchy obtained by choosing any (finite) graph, with any rate (i.e. input/output flow) and any cost function in the class.*

We will assume that $\mathcal{C}$ *contains the constant functions. Moreover we assume that* $\mathcal{C}$ *contains only non-decreasing functions.*

1. *What is the price of anarchy if* $\mathcal{C}$ *is the class of non-decreasing affine functions?*

2. *What is the price of anarchy if* $\mathcal{C}$ *is the class of non-decreasing polynomial functions?*

A Pigou network is a network with two nodes: one origin o, *one destination* d, *and two arcs linking* o *to* d, *and a flow rate between* o *and* d *of* $r > 0$. *The cost function of the first arc is* $c \in \mathcal{C}$, *and the cost function of the second arc is constant equal to* $c(r)$.

3. *What is the user equilibrium, social optimum and price of anarchy of a Pigou network with given rate* $r > 0$? *(The result is not a closed formula but contains a max or a min).*

4. *Deduce a lower bound of the price of anarchy of the class* $\mathcal{C}$.

5. *Show that this lower bound is exact for the class of affine non-decreasing functions.*

Solution. 1. *(0.5pt) By theorem it is lower than* $4/3$, *and the bound is attained for the Braess Paradox.*

2. *(1pt)* $+\infty$ *using* $r = 1$ *and* x^p, $p \to \infty$.

3.
   - *(1pt) By monotonicity of* c, $c(x) \leq c(r)$ *for* $x \leq r$, *hence an user equilibrium is given by* $x_1 = r$, $x_2 = 0$, *with total cost* $rc(r)$.
   - *(1pt) A social optimum is given by* $\min_{0 \leq x \leq r} xc(x) + (r-x)c(r)$
   - *(0.5pt) The price of anarchy* $\max_{0 \leq x \leq r} rc(r)/(xc(x) + (r-x)c(r))$.

4. *(1pt) Thus the price of anarchy of $\mathcal{C}$ is greater than the price of anarchy on a Pigou network, hence greater than*

$$\max_{c \in \mathcal{C}, r>0} \max_{0 \le x \le r} \frac{rc(r)}{xc(x) + (r-x)c(r)}.$$

5. *(1pt) Choosing $r = 1$ and $c(x) = x$ yield a lower bound of $4/3$ which is exact.*

Exercice 5 (bonus)**.** *Give 3 optimization problems tackled by Air France-KLM Operations Research group.*