

ENPC - Operations Research and Transport - 2018

You have 2.5 hours for the exam. Exercises are independent. Computer, phones, tablets and every connected objects are forbidden. Every note is allowed.

Exercise 1 (2pts). Consider a game where rewards (to be maximized) are given by the following table where actions of player 1 correspond to the lines, actions of player 2 to the columns, rewards being given in the order of player. For example, if player 1 play a, and player 2 play c, then player 1 gains 2 and player 2 gains 3.

	a	b	c
a	(7,1)	(0,0)	(2,3)
b	(-1,2)	(2,3)	(3,2)
c	(-1,4)	(1,3)	(1,7)

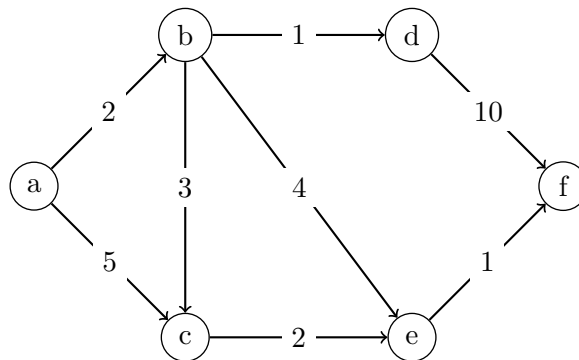
1. Find the Nash equilibrium(s)
2. Find the social optimum(s)
3. Find the Pareto optimum(s)

Solution. 1. (0.5pt) NE : (b,b)

2. (0.5pt) SO : (a,a), (c,c)

3. (1.25 pt) Pareto : (a,a), (a,c), (b,b), (b,c), (c,c),

Exercise 2 (5pts). Consider the following weighted graph.



1. Use Dijkstra's algorithm to find the cost of the shortest path between node a and node f. The results can be presented in a table of the labels where each column corresponds to a node of the graph, and each line to an iteration of the Dijkstra algorithm. Note the order in which the nodes are treated.

a	b	c	d	e	f
20	4	7	6	0	0

2. We have the following heuristic h giving an estimate of the distance between a given node and f .

Apply the A^* algorithm using this heuristic. Note the order of nodes treated. Comment.

Solution. 1. We have (2 pt)

a	b	c	d	e	f
(0)	∞	∞	∞	∞	∞
0	(2)	(5)	∞	∞	∞
0	2	(5)	(3)	(6)	∞
0	2	(5)	3	(6)	(13)
0	2	5	3	(6)	(13)
0	2	5	3	6	(7)
0	2	5	3	6	7

Hence the shortest path from a to f as cost 7. The nodes are treated in the following order: $a-b-d-c-e-f$.

2. (3pt)

We compute the label λ in the following table

a	b	c	d	e	f
(20)	∞	∞	∞	∞	∞
20	(6)	(12)	∞	∞	∞
20	6	(12)	(9)	(6)	∞
20	6	(12)	(9)	6	(7)
20	6	(12)	(9)	6	7

Which gives the distance and shortest path in only 4 iterations ($a-b-e-f$) instead of 6.

Exercice 3 (8pts). Consider a (finite) directed, strongly connected, graph $G = (V, E)$. We consider K origin-destination vertex pair $\{o^k, d^k\}_{k \in \llbracket 1, K \rrbracket}$. We denote by (G, ℓ, r) the congestion game where

- r^k is the intensity of the flow of users entering in o^k and exiting in d^k ;
- \mathcal{P}_k is the set of all simple (i.e. without cycle) paths from o^k to d^k , and by $\mathcal{P} = \bigcup_{k=1}^K \mathcal{P}_k$;
- f_p the number of users taking path $p \in \mathcal{P}$ per hour (intensity);
- $f = \{f_p\}_{p \in \mathcal{P}}$ the vector of path intensity;
- $x_e = \sum_{p \ni e} f_p$ the flux of user taking the arc $e \in E$;
- $x = \{x_e\}_{e \in E}$ the vector of arc intensity;
- $x(f)$ is the vector of edge-intensity induced by the path intensity f ;
- $\ell_e : \mathbb{R} \rightarrow \mathbb{R}^+$ the cost incurred by a given user to take edge e , if the edge-intensity is x_e ;

- $L_e(x_e) := \int_0^{x_e} \ell_e(u) du.$

We want to compare the cost of the user equilibrium of (G, ℓ, r) , denoted $f^{UE,r}$, with the cost of the social optimum $f^{SO,2r}$ of $(G, \ell, 2r)$, that is the same game with twice the inflows. Accordingly we denote $x^{UE,r} = x(f^{UE,r})$, and $x^{SO,2r} = x(f^{SO,2r})$. Finally, edge-loss ℓ_e are assumed to be non-negative and non-decreasing.

We construct new loss functions $\bar{\ell}_e(x)$ given by

$$\bar{\ell}_e(x) = \begin{cases} \ell_e(x^{UE,r}) & \text{if } x \leq x^{UE,r} \\ \ell_e(x) & \text{else} \end{cases}$$

Accordingly we denote $\bar{\ell}_p(f) = \sum_{e \in p} \bar{\ell}_e(x_e(f))$ and

$$C(x) = \sum_{e \in E} x_e \ell_e(x_e) \quad \text{and} \quad \bar{C}(x) = \sum_{e \in E} x_e \bar{\ell}_e(x_e).$$

1. Justify that for all $k \in \llbracket 1, K \rrbracket$, there exists $c_k \in \mathbb{R}_+$ such that for all path $p \in \mathcal{P}_k$,

$$f_p^{UE,r} > 0 \Rightarrow \ell_p(f^{UE,r}) = c_k.$$

2. Show that, for any $x \in \mathbb{R}_+^{|E|}$, $C(x) \leq \bar{C}(x)$, and that $C(x^{UE,r}) = \bar{C}(x^{UE,r})$.

3. Show that, for any $x \in \mathbb{R}_+^{|E|}$, $x_e(\bar{\ell}_e(x_e) - \ell_e(x_e)) \leq x_e^{UE,r} \ell_e(x_e^{UE,r})$.

4. Deduce that, $\bar{C}(x^{SO,2r}) - C(x^{SO,2r}) \leq C(x^{UE,r})$.

5. On the other hand, show that, for every path $p \in \mathcal{P}_k$, $\bar{\ell}_p(f^{SO,2r}) \geq c_k$.

6. Write C and \bar{C} as function of f instead of x (we keep the same notation).

7. Deduce that, $\bar{C}(f^{SO,2r}) \geq 2C(f^{UE,r})$.

8. Finally, show that, $C(f^{UE,r}) \leq C(f^{SO,2r})$. Give an interpretation of this result.

Solution. 1. (0.5pts) $f^{UE,r}$ is a Wardrop equilibrium, thus by definition the cost of all used path is the same.

2. (0.5pts) As ℓ_e are non decreasing, $\bar{\ell}_e \geq \ell_e$, multiplying by $x_e \geq 0$ and summing gives the result. Equality is obvious.

3. (1pts) $\bar{\ell}_e(x_e) - \ell_e(x_e)$ is null if $x_e \geq x_e^{UE,r}$, and equal to $\ell_e(x_e^{UE,r}) - \ell_e(x_e) \leq \ell_e(x_e^{UE,r})$ otherwise. Multiplying by x_e we have the result both for $x_e \geq x_e^{UE,r}$ and for $x_e \leq x_e^{UE,r}$.

4. (1pts)

$$\begin{aligned} \bar{C}(x^{SO,2r}) - C(x^{SO,2r}) &= \sum_{e \in E} x_e^{SO,2r} (\bar{\ell}_e(x_e^{SO,2r}) - \ell_e(x_e^{SO,2r})) \\ &\leq \sum_{e \in E} x_e^{UE,r} \ell_e(x_e^{UE,r}) \\ &= C(x^{UE,r}) \end{aligned}$$

5. (1.5pts) Consider $p \in \mathcal{P}^k$. Then $\ell_p(f^{UE,r}) = c_k$. Furthermore,

$$\bar{\ell}_p(f^{SO,2r}) = \sum_{e \in E} \bar{\ell}_e(x_e(f^{SO,2r})) \geq \sum_{e \in E} \ell_e(x_e^{UE,r}) = c_k$$

where the inequality comes from monotonicity of ℓ_e , and definition of $\bar{\ell}_e$.

6. (0.5pts)

$$C(x) = \sum_{f \in \mathcal{P}} f_p \ell_p(f) \quad \text{and} \quad \bar{C}(x) = \sum_{f \in \mathcal{P}} f_p \bar{\ell}_p(f).$$

7. (2pts)

$$\begin{aligned} \bar{C}(f^{SO,2r}) &= \sum_{k=1}^K \sum_{p \in \mathcal{P}_k} f_p^{SO,2r} \bar{\ell}_p(f^{SO,2r}) \\ &\geq \sum_{k=1}^K c_k \sum_{p \in \mathcal{P}_k} f_p^{SO,2r} \\ &= \sum_{k=1}^K 2c_k r^k \\ &= 2C(f^{UE,r}) \end{aligned}$$

8. (1pts) Combining previous results we have

$$2C(f^{UE,r}) \leq \bar{C}(x^{SO,2r}) \leq C(x^{UE,r}) + C(x^{SO,2r}),$$

which give the result, that can be interpreted as "optimizing flux cannot allow more than twice the inflows rates without increasing global cost".

Exercise 4 (7pts). Consider the function $f(x_1, x_2) = 4x_1^4 - 2x_1 + x_2^2 - x_2 + 2$, and the set

$$X = \{x \in \mathbb{R}_+^2 \mid 2x_2 + x_1 \leq 2\}$$

and $x^0 = (0, 0)$. A scheme of X representing the iteration and search direction of the algorithm might be helpful.

1. Justify that X is polyhedral and find its extreme points.
2. Compute ∇f
3. Justify that this problem can be solved by Frank-Wolfe (aka conditional gradient) algorithm.
4. Find the descent direction d^0 of the Frank-Wolfe algorithm starting from x_0 .
(hint : use the extreme points of X).
5. Find the optimal step t^0 of the first step of Frank-Wolfe algorithm. What is the new point x^1 ?
6. What is the upper and lower bound obtained along this first iteration ?
7. Find the descent direction d^1 of the second step of Frank-Wolfe algorithm.

8. Write the unidimensional optimisation problem that would determine the next optimal step t^1 (do not solve it).
9. Compute the lower bound associated to the second step of the algorithm.

Solution. 1. (0.5pts) $(0, 0)$, $(0, 1)$ and $(2, 0)$

2. (0.5pts) $\nabla(f) = \begin{pmatrix} 16x_1^3 - 2 \\ 2x_2 - 1 \end{pmatrix}$

3. (0.5pts) $\nabla^2 f(x) = \begin{pmatrix} 48x_1^2 & 0 \\ 0 & 2 \end{pmatrix} \succeq 0$ hence f is convex, and X is polyhedral and bounded.

4. (1pts) $\min_{y \in X} -2y_1 - 1y_2$. $-2 * 0 - 1 * 1 > -2 * 2 - 1 * 0$, hence the optimal solution is $y^0 = (2, 0)$. And the optimal direction is $d^0 = y^0 - x^0 = (2, 0)$.

5. (1pts) $\min_{t \in [0,1]} 2^6 t^4 - 2^2 t$. By derivating this objective function we obtain that the optimal step is $t^0 = 1/4$, and $x^1 = x^0 + t^0 d^0 = (1/2, 0)$.

6. (1pts) The upper bound is $f(x^1) = 5/4$. The lower bound is $(\nabla(f)(x^0))^T (y^0 - x^0) + f(x^0) = -2$.

7. (1pts) To find the direction d^1 we need to solve $\min_{y \in X} \nabla(f)(x^1)^T y$. The solution being an extreme point of X that is neither $(0, 0)$ nor $(2, 0)$, we have $y^1 = (1, 0)$. Thus $d^1 = (1, -1/2)$.

8. (0.5pts) Finding the optimal step require to solve

$$\min_{t \in [0,1]} 4(1/2 - t)^4 - 2(1/2 - t) + t^2 - t$$

9. (1pts) The lower bound is given by

$$(\nabla(f)(x^1))^T (y^1 - x^1) + f(x^1) = 0 * (-1/2) - 1 * 1 + 5/4 = 1/4$$

Exercise 5 (Bonus). According to Yuso, why is the package transport problem more complicated than the taxi problem ? Why are Yuso solving shortest path problem for ?