1. INTRODUCTION

A remarkable model of economic growth and the class struggle has been presented by Goodwin (1967). The main insight of this «Marxian» or neo-Keynesian model is that, when capital and labour are complementary factors of production, workers and capitalists have a symbiotic relationship and as a consequence are in perpetual conflict with each other. Whenever profitability improves it carries the seed of its own destruction by engendering a too vigorous expansion of output and employment, thus destroying the reserve army of labour and strengthening labour’s bargaining power (Goodwin, 1967, p. 58). On the other hand, there is the famous neo-classical model of economic growth which relies on smooth substitution between the factors of production and capitalists and workers living in splendid harmony (Solow, 1956). The economy is, however, characterized by neither strictly complementary factors nor perfect substitution between factors. The main objective of this paper is to derive an appropriate theory which captures the essential features of both the neo-Keynesian and the neo-classical views and therefore allows a deeper understanding of the debate between the two camps.

In order to shed some light on these issues it is useful to recollect Goodwin’s original premises. The following five assumptions were made for convenience: (1) steady disembodied technical progress; (2) steady growth in the labour force; (3) only two factors of production, labour and «capital», both homogeneous and non-specific; (4) all quantities
real and net; (5) all wages consumed, all profits saved and invested (1).
There were two further assumption of a more empirical, and disputable, nature: (6) a constant capital-output ratio; (7) a real wage rate which rises in the neighbourhood of full employment (Goodwin, 1967, p. 54). In previous work (6) has been relaxed by allowing entrepreneurs to choose cost-minimizing directions of labour-saving vs. capital-saving innovations based upon the Kennedy-Weizsäcker invention possibility frontier or Kaldor's technical progress function (Shah and Desai, 1981; van der Ploeg 1983). Such «learning by doing effects» lead to economic growth and damped conflict cycles. This paper is mainly concerned with the relationship between neo-Keynesian and neo-classical theory, hence (6) is relaxed by having firms maximize profits subject to a C.E.S. production function. The advantage of this relaxation of (6) is that both the Leontief technology underlying Goodwin's model and the more general technology underlying Solow's (1956) model are incorporated as special cases. At the same time (7) is replaced by the assumption that when the bargaining strength of the workers' movement, influenced by the employment rate and growth in the productivity of labour, is high then the share of labour in value added rises. The case of perfect compensation for productivity growth in real wage bargaining and an infinite speed of clearing the labour market corresponds closest to the neo-classical model. It will become clear that assumptions (6) and (7) of Goodwin (1967) lead to very special results.

2. A CLASSICAL MODEL OF ECONOMIC GROWTH AND THE DISTRIBUTION OF INCOME

Let \( q, k, l, n, w, a = q/l \) and \( \sigma = k/q \) denote real output, capital, employment, the labour supply, the real wage rate, labour productivity and the capital-output ratio, respectively (\(^2\)). Steady disembodied technical progress is captured by measuring labour input in efficiency units, that is \( e = \exp(\alpha t)l \) where \( \alpha \) is the Harrod-neutral rate of labour-augmenting technical progress. The labour force grows at the steady rate \( \beta \), so that

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(1) With respect to (1), Solow (1960) provides a putty-putty vintage model of embodied technical progress based upon the Cobb-Douglas production function and Johanson (1959) provides a putty-clay model of embodied technical progress. However, these studies do not discuss aspects of the class struggle. Stone (1984) relaxes (2) to allow for saturation but one could also envisage extending (2) to allow for Malthusian effects, Desai (1973) and van der Ploeg (1983) relax (4) and van der Ploeg (1984) relaxes (5).

(2) The notation conforms to Goodwin (1967).
$n = \exp(\beta t) n_0$. All variables are in real terms and net of depreciation. The classical savings hypothesis, $k = \{1 - (w/a)\} q$, implies that capitalists save all their profits income and use this internal finance entirely for investment purposes, whereas workers consume all their wages. In this model firms (capitalists) are liquidity constrained as the only other agents (workers) do not save, so investment cannot be financed by borrowing.

So far, the model incorporates assumptions (1)-(5) of Goodwin (1967). The sixth assumption of a constant capital-output ratio, $\sigma$ fixed, is replaced by the constant returns to scale C.E.S. production function,

$$q = F(k, e) = e^{\mu k^{-\delta} + (1 - \mu) e^{-\delta}}^{-1/\delta}, 0 < \mu < 1, \delta > 0,$$

and the profit-maximizing hypothesis, $F_e = w \exp(-\alpha t)$. The C.E.S. production function has a constant elasticity of substitution given by $\varepsilon = d \log(k/e)/d \log(\theta) = 1/(1 + \delta)$, where $\theta = F_e/F_k$. Note that as $\delta \to 0 (\varepsilon \to 1)$, the C.E.S. production function is replaced by the Cobb-Douglas production function, $q = ck^\mu e^{-\mu}$; and as $\delta \to \infty (\varepsilon \to 0)$, (1) becomes the Leontief technology, $q = \min(ck, ce)$, so that $\sigma = 1/c$. The first case corresponds to perfect substitution between capital and labour whereas the latter case corresponds to complementary factors. Firms hire labour until the productivity of the marginal worker equals the real wage which, using the intensive form of (1), yields the optimal factor demand ratio,

$$\kappa(u) = \frac{k}{e} = \left\{\frac{(1 - \mu)(1 - w)}{\mu w}\right\}^{-1/\delta}, \kappa' > 0$$

where $u = w/a$ denotes the workers' share of the national income. It follows that the profit-maximizing capital-output ratio is given by

$$\sigma(u) = \frac{1}{c} \left(\frac{1 - u}{\mu}\right)^{-1/\delta}, \sigma' > 0$$

and the productivity of labour by

$$a(u) = c \left(\frac{u}{1 - \mu}\right)^{1/\delta} \exp(\alpha t), \quad a' > 0.$$

When workers manage to obtain a large share of the product, firms find it less profitable to hire workers and therefore switch away from labour to machinery.
Finally, assumption (7) is replaced by the bargaining equation

\[
\frac{\dot{w}}{w} = f(v) + \varrho \frac{\varrho}{\varrho}, \quad f' > 0, \quad f'' > 0, \quad \varrho > 0
\]  

(5)

where \( v = l/n \) denotes the employment rate. Rationing of the capitalists’ demand for labour is avoided with the assumption \( f' \to \infty \) as \( v \to 1 \). Downward rigidity of real wages may be allowed for by having \( f' \to 0 \) for low values of \( v \). The bargaining equation allows for the observation that when the reserve army of unemployed, \( 1 - u \), is small or labour productivity growth is high, the workers become more powerful and are able to press for higher real wages. The productivity term may be interpreted as an «ability-to-pay» hypothesis. Observe that Goodwin (1967) corresponds to \( \varrho = 0 \), but even if productivity growth affects the wage bargain in Goodwin’s model, this does not matter as it is exogeneous in his model (due to assumption (6)). From (4) it follows that, as far as the profit-maximizing behaviour of firms is concerned, \( (1/\varrho)\% \) is the maximum permitted increase in the growth of real wages after workers have allowed an increase of \( 1\% \) in the growth of labour productivity. This puts an upper bound on the compensation for productivity improvements the labour movement can achieve (\( \varrho < 1/\varrho \)).

3. EQUILIBRIUM AND TRANSIENT DYNAMICS

The classical growth cycle model described above can be cast into a two-dimensional state-space model in terms of the share of labour, \( u \), and the employment rate, \( v \):

\[
\frac{\dot{u}}{u} = \{f(v) - (1 - \varrho)\alpha]/[1 + (1 - \varrho)/\delta] = \tilde{f}(v); \tag{6}
\]

\[
\frac{\dot{v}}{v} = c\mu^{-1/\delta}(1 - u)^{1/\delta} - \left( \frac{1}{\delta(1 - u)} \right) \frac{\dot{u}}{u} - (\alpha + \beta). \tag{7}
\]

The singular points (equilibrium) follow from \( \dot{u} = \dot{v} = 0 \). The equilibrium employment rate is given by

\[
v^* = f^{-1}((1 - \varrho)\alpha) \tag{8}
\]

and increases with uncompensated labour-augmenting technical progress.
The equilibrium share of labour in value added is equal to

\[ u^* = 1 - \left( \frac{\alpha + \beta}{c} \right)^{1-\epsilon} \mu^\epsilon \]  

which becomes \( u^* = 1 - \mu \) for the Cobb-Douglas case \((\epsilon \to 1)\) and \( u^* = 1 - (\alpha + \beta)/c \) for the Leontief technology \((\epsilon \to 0)\). For the general C.E.S. technology, the equilibrium share of wages in the national income is a decreasing function of the natural rate of growth, \( \alpha + \beta \), the extent of which diminishes with the elasticity of substitution, \( \varepsilon \). In the dynamic equilibrium of this classical model, both the growth rate in output (and the capital stock) and the profit rate must equal the natural rate of growth, \( \alpha + \beta \). In equilibrium the growth in real wages matches the growth in labour productivity.

**Theorem 1.** The dynamics of an economy with C.E.S. production, defined by (6)-(7), in the neighbourhood of the equilibrium (8)-(9) corresponds to a stable spiral or stable node provided that \( 0 < \varepsilon < 1 \) (\( \delta \) finite) and \( \varrho < 1/\varepsilon \). The degenerate case \( \varrho = 1/\varepsilon \) corresponds to a stable node (see fig. 1(a)).

![Figure 1. Phase-portraits of classical growth cycles](image)

(a) C.E.S. technology \((0 < \varepsilon < 1)\)  
(b) Leontief technology \((\varepsilon = 0)\)

**Proof.** The Jacobian of the system (6)-(7) evaluated at (8)-(9), say \( J^* \), has

\[ \theta_1 = \text{trace}(J^*) = -\left( \frac{\epsilon}{1-u^*} \right) \left( \frac{f'(v^*) \cdot v^*}{1 - \varrho \varepsilon} \right) \]  

and

\[ \theta_0 = \text{det}(J^*) = cu^* \left( \frac{1-u^*}{\mu} \right)^{1/\delta} \left( \frac{f'(v^*) \cdot v^*}{1 - \varrho \varepsilon} \right) \]
Since $0_1 < 0$, $0_2 > 0$ and the system (6)-(7) is assumed to be only moderately nonlinear (e.g., Aggarwal, 1972, p. 25), we have established Poincaré (local) stability of the equilibrium (8)-(9). The discriminant of the characteristic polynomial $\lambda^2 - 0_1\lambda + 0_2 = 0$ may be written as

$$\Delta = 0_1^2 - 40_2 =$$

$$= \left(f'(v^*)v^* - 4cu^* \left(\frac{1 - u^*}{\mu}\right)^{1/\delta} (1 - u^*)^2 (1 - 0_2)/\varepsilon^2\right) / 0_3$$

(12)

where $0_3 = (1 - \delta)^2 (1 - u^*)^2/(f'(v^*)v^*) > 0$. It follows that for large (small) enough $f'(v^*)$ or $0_2$, $\Delta$ will be positive (negative) which implies that the dynamics correspond to a stable node (spiral). For the degenerate case $v = f^{-1}(\delta x) = v^2$ and

$$\dot{u} = cu^{-1/\delta} \delta(1 - u)^{1+2\delta/\delta} u - \delta(1 - u)(\alpha + \beta)u \simeq -$$

$$- [(\alpha + \beta)/\varepsilon] u^*(u - u^*)$$

(13)

which gives a Poincaré stable node.

The case of a Leontief technology ($\varepsilon \to 0$) leads to $0_1 \to 0$ and $0_2 \to -cu^*f'(v^*)v^*$, which gives imaginary roots ($\lambda_1, \lambda_2 = \pm i\sqrt{0_2}$) and one can therefore not infer anything about the local dynamics of the nonlinear system (6)-(7) from the linearized system (e.g., Aggarwal, 1972, p. 26).

**Theorem 2.** The equilibrium of an economy with a Leontief technology ($\varepsilon = 0$), defined by (6) and

$$\frac{\dot{v}}{v} = c(1 - u) - (\alpha + \beta),$$

(7')

is a center. The corresponding phase-plane portrait yields closed orbits, where an approximation to the period of the cycles is given by

$$T \simeq 2\pi \left[\{c - (\alpha + \beta)\}f' \{f^{-1}[(1 - \delta)\alpha]\}f^{-1}[(1 - \delta)\alpha]^{-1}\right]$$

(14)

Furthermore, the orbits generated by (6) and (7') are Lyapunov stable, asymptotically unstable and structurally unstable (see fig. 1(b)).

**Proof.** The function

$$L(u, v) = c[u - u^* \ln(u)] + \int_0^1 f^{-1}(v) \, dv$$

(15)
satisfies $L(u, v) = 0$ on the solution trajectories of (6) and (7'), so that the system is conservative. Since $u^*$ and $v^*$ define the absolute minimum of $L(u, v)$, $L(u, v)$ is a Lyapunov function and the system is Lyapunov stable. It can be shown that the system has an integral variant, so that the solution orbits are closed. Since the orbits are not limit cycles, the system is asymptotically unstable. The equilibrium of (6) and (7') is not hyperbolic, as the roots of the characteristic polynomial have zero real parts, so that the system is structurally unstable (further details are in Velupillai (1979)). Expression (44) follows immediately from $T \cong 2\pi i/\lambda_{1,2} = 2\pi \sqrt{\theta}$. □

Theorem 2 summarized, of course, the main properties of a Goodwin-type/Volterra-Lotka model of perpetual conflict between capitalists and workers (see introduction). The structural instability property of the Goodwin model is important, since a minor modification in the parameters of the Goodwin model can lead to a radical change in the quantitative behaviour of the economic model. For example, if there is a small perturbation in the elasticity of substitution ($\varepsilon \cong 0$), the phase-portrait changes from a center to a stable focus (see Theorem 1).

The case of a Cobb-Douglas technology ($\varepsilon = 1$) leads to $u = 1 - \lambda$, 

$$\dot{x} = f(v)/(1 - \eta) - (\alpha/\mu)$$  \hspace{1cm} (6')

and

$$\dot{v} = c + \mu u^{-1} - (\alpha/\mu) - (\alpha + \beta)$$  \hspace{1cm} (7')

so that the state-space variables are now $x$ and $v$. The equilibrium is defined by (8) and $\sigma^* = \lambda/\alpha + \beta$ or

$$x^* = \left(\frac{c}{\alpha + \beta}\right)^{1/(1-\mu)}$$  \hspace{1cm} (16)

Hence, a high rate of natural growth requires relatively less machinery in equilibrium.

**Theorem 3.** The dynamics of an economy with Cobb-Douglas production ($\varepsilon = 1$) defined by (6') and (7'), in the neighbourhood of the equilibrium (8) and (16) corresponds to a stable spiral or stable node provided
that $q < 1$. The degenerate case $q = 1$ corresponds to a node, which is uniformly asymptotically stable in the large.

Proof. The Jacobian of the system (6') and (7') evaluated at (8) and (16), say $J^*$, has

$$\theta_1 = \text{trace}(J^*) = -f'(v^*)v^*/\{(1-q)\lambda\} \quad (10')$$

and

$$\theta_0 = \text{det}(J^*) = (\alpha + \beta)(1-\lambda)f'(v^*)v^*/\{(1-q)\lambda\} \quad (11')$$

The equilibrium (8) and (16) is Poincaré stable because $\theta_1 < 0$ and $\theta_0 > 0$. The discriminant is given by

$$A = (f'(v^*)v^* - 4(\alpha + \beta)(1-\lambda)/(\lambda(1-q)))/\theta_3 \quad (12')$$

whose sign is indeterminate. The degenerate case $q = 1$ gives rise to $v = f^{-1}(0) = v^*$ and the Bernoulli equation

$$\dot{x}/x = c\lambda x^{\lambda-1} - (\alpha + \beta) \quad (17)$$

The solution to (17) is

$$x(t) = \{c[\sigma^* + (\sigma_0 - \sigma^*) \exp(\theta_0 t/\theta_1)]\}^{1/(1-\lambda)} \quad (18)$$

which is uniformly asymptotically stable in the large as

$$\theta_0/\theta_1 = -(\alpha + \beta)(1-\lambda) < 0. \quad \square$$

4. CONCLUSIONS

When the assumption of a fixed capital-output ratio is replaced by the assumption that firms maximize profits subject to a C.E.S. production function, whilst retaining the wage-bargaining equation of Goodwin (1967), the perpetual class struggle cycles of Goodwin's model are replaced by either damped conflict cycles or monotonic convergence to the balanced growth equilibrium. Improved profitability still carries the seed of its own destruction by stimulating investment, output, jobs, workers' bargaining strength and therefore real wages, but in addition to this internal finance argument firms substitute by firing some workers and installing
some extra capital over the course of a conflict cycle. It is this additional factor of "competition within the species" that destroys the conservative nature of the Goodwin system and gives rise to the asymptotically stable trajectories of Theorem 1 and 3. A small elasticity of substitution, a relatively small effect of the reserve army of unemployed upon workers' bargaining strength or incomplete compensation for productivity increases are more likely to lead to conflict cycles than to monotonic convergence to the balanced growth trajectories. Indeed, the species case where workers manage to obtain the maximum possible compensation for increases in labour productivity \( \rho = 1/\sigma \) never leads to class struggle cycles and is perhaps most akin to the neoclassical text-book model of economic growth (Solow, 1956; Allen, 1967, Chapter 14), because the labour market then clear instantaneously and the employment rate is always maintained at a constant (natural) level.

The special case of no substitution possibilities leads to a Leontief technology and the Goodwin model, where the period of the perpetual conflict cycles increases with the equilibrium share of profits (which increases with the capital-output ratio and the natural rate of economic growth) and diminishes with the uncompensated growth in labour-augmenting technical progress. Even when there is maximum scope for substitution possibilities (as with the Cobb-Douglas technology), the possibility of damped conflict cycles exists, despite the fact that the functional distribution of income and therefore the aggregate savings ratio no longer vary when the economy is in disequilibrium.

The growth cycles discussed in this paper have been thoroughly classical in the sense that Say's law has been strictly imposed. A natural further relaxation of assumption 6 of Goodwin (1967) would be to allow natural and warranted growth rates to differ by introducing an independent investment function. Such a function would depend on expectations of future effective demand and profitability as well as the internal finance motives discussed in this paper. The advantage of this further extension would be to allow for more realistic features such as excess capacity and Keynesian (as well as classical) unemployment. With respect to assumption 7 of Goodwin (1967), it can be shown that under-compensation for increases in the cost of living ("money illusion"), adaptive expectations rather than perfect foresight and fear of redundancies ("loops" in wage formation) also transform the perpetual conflict cycles of Goodwin (1967) into damped conflict cycles.

These provide further examples, based upon "perturbing" assumption 7 rather than assumption 6, of the structural instability of Goodwin's
model. It is clear that a more realistic theory combining both classical and Keynesian elements should also pay more attention to the determinants of inflation and the consequences for aggregate supply.

REFERENCES