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Champs de déplacements et de contraintes autour des ouvrages profonds

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Septembre 2017

1.1 Introduction

Le problème classique de déformation d'un corps solide se présente sous la forme suivante : considérons un corps solide déformable occupant un domaine Ω dans l'espace, de frontière $\partial\Omega$. On part d'un état libre de contraintes qui sert d'origine des déplacements pour les points matériels. On soumet ce corps à des forces volumiques $\underline{F}$ agissant dans Ω, des forces surfaciques $\underline{f}$ agissant sur une partie du contour notée $\partial_f\Omega$, tandis que les déplacements $\underline{u}^d$ sont imposés sur la partie du contour $\partial_u\Omega$ complémentaire du $\partial_f\Omega$. Notons que le partage entre $\partial_f\Omega$ et $\partial_u\Omega$ peut être différente pour les différentes degrés de liberté de forces et de déplacements. Pour un problème posé, en chaque point du contour $\partial\Omega$, 3 degrés de libertés parmi les 6 degrés de libertés de forces et de déplacements sont imposé en évitant d'imposer une composante du déplacement et une composante de force dans la même direction. On peut imposer, par exemple, (u_x, u_y, f_z) ou (u_x, f_y, f_z) sur une partie de $\partial\Omega$. Sous l'effet de ces forces et conditions aux limites imposées, le corps se déforme et les déformations induisent des contraintes suivant la loi de comportement ou le modèle du matériau constitutif. Les contraintes doivent respecter les équations d'équilibre.

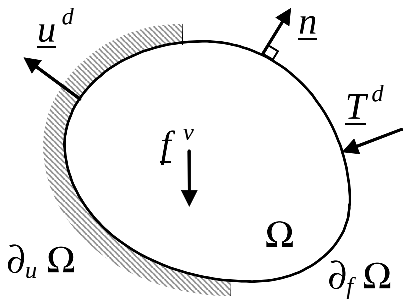


Figure - : Problème aux limites du corps solide déformable Ω soumis à des forces volumiques $\underline{F}$ dans Ω, aux forces surfaciques $\underline{f}$ sur la partie $\partial_f\Omega$ du contour et aux déplacements $\underline{u}^d$ sur la partie complémentaire $\partial_u\Omega$.

Les équations de ce problème sont les suivantes :

Equations d'équilibre des contraintes:

(E1): $\forall \underline{x} \in \Omega$; $\quad$ $div\, \boldsymbol{\sigma}\, (\underline{x}) + \underline{F}(\underline{x}) = 0$

(E2): $\forall\ \underline{x} \partial\in\ \partial_f\Omega$; $\quad \boldsymbol{\sigma}.\underline{n} = \underline{f}$,

Conditions aux limites en déplacements:

(E3): $\forall\ \underline{x} \partial\in \partial_u\Omega$; $\quad \underline{u}\ (\underline{x}) = \underline{u}^d\ (\underline{x})$

A ces relations il faut ajouter la loi du matériau constitutif de Ω reliant la contrainte à la déformation, $\boldsymbol{\sigma} = \mathcal{F}(\boldsymbol{\varepsilon})$, où :

$$\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla \underline{u} + {}^t\nabla \underline{u})$$

La voie la plus simple pour résoudre ce problème, surtout dans le cas de comportements non linéaires est de prendre pour inconnue principale le champ de déplacements $\underline{u}(\underline{x})$. En calculant à partir de ce champ de déplacement, successivement, la déformation et la contrainte, et en reportant dans les équations d'équilibre ci -dessus, on trouvera un système d'équations portant sur le champ $\underline{u}$. Dans le cas d'un comportement élastique linéaire de tenseur d'élasticité $\mathbb{C}$, ces équations s'écrivent :

(E$_1$) : $\forall\ \underline{x} \in \Omega\ ;\quad C_{ijkl}\partial_{jk}u_l(\underline{x}) + F_i(\underline{x}) = 0$

(E$_2$) : $\forall\ \underline{x} \in \partial_f\Omega\ ;\ n_j(\underline{x})\ C_{ijkl}\ \partial_k u_l(\underline{x}) = f_i(\underline{x})$

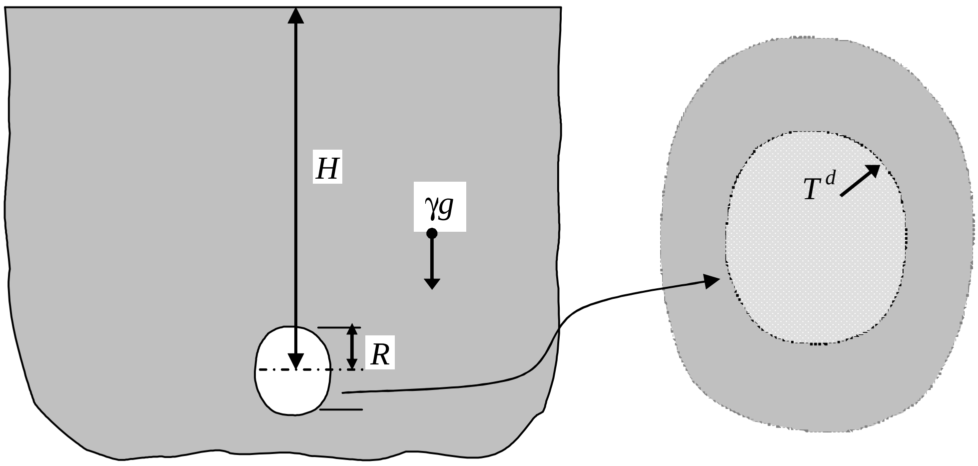
(E$_3$) : $\forall\ \underline{x} \in \partial_u\Omega\ ;\ \underline{u}(\underline{x}) = \underline{u}^d(\underline{x})$

Il peut être démontré que, pour un problème bien posé (complémentarité $\partial_f\Omega$ et $\partial_u\Omega$ indiquée ci-dessus), et la défini-positivité du tenseur $\mathbb{C}$ (propriété admise pour les matériaux élastiques), le système d'équations ci-dessus à une solution unique, éventuellement à un déplacement de corps rigide près qui peut être supprimé avec un choix adéquate de $\underline{u}^d$. Mais sauf pour des cas exceptionnels de géométries très simples de Ω et des distributions de forces très particulières, ce problème ne peut être résolu analytiquement et sa résolution repose en général sur des méthodes numériques.

Dans la suite, nous allons considérer des cas de géométries simples, sphériques ou cylindriques, où des solutions analytiques peuvent être établies pour ces équations. Ces solutions seront utiles pour l'étude des ouvrages souterrains profonds. Mais pour ce faire, nous devons introduire d'abord quelques hypothèses simplificatrices qui permettent de passer d'un problème pesant (présence de forces volumiques) à un problème non pesant.

1.2 Transformation of the problem to a problem without volume forces

As mentioned above, for both analytical and numerical methods, existence of body forces leads to more complex equations to be solved. Therefore, it is costumary to find a method to remove the volume forces from the problem and replace it by appropriate boundary forces. For underground openings, body forces are intimately related to initial stress field. The displacement and strains are computed from the initial state with a stress $\boldsymbol{\sigma}^0$. In the following we analyze the exact problem formulated with body forces and show what are the successive approximations and simplifications for special cases that allow the transformation to problem without body forces.



Before creation of the cavity the forces applied by the fill materials on the boundary of the future cavity's wall are:

$$\underline{T}^{0d} = \boldsymbol{\sigma}_0(\underline{z}).\underline{n} \tag{1.1}$$

Where the $\boldsymbol{\sigma}_0(\underline{z})$ is the ground stress field which in equilibrium with the volumetric forces that are the self-weight:

$$div\, \boldsymbol{\sigma}_0 + \underline{\gamma}^{0d} = 0 \tag{1.2}$$

The excavation action consists first in removing the cavity's space from modelling and impose the surface traction forces $-\underline{T}^{0d} = -\boldsymbol{\sigma}_0(\underline{z}).\underline{n}$ on the boundary of the hole that remains in the formation. The total traction on this boundary will then be equal to zero. This represents a traction free boundary and corresponds to an empty cavity. This condition must be indeed completed by application of a pressure corresponding to fluids filling the cavity. Under the excavation action, the formation rocks undergo a displacement $\underline{u}$ resulting in a strain $\boldsymbol{\varepsilon}$. If the formation material can undergo irreversible deformation (plastic, viscous, damage…) denoted by $\boldsymbol{\varepsilon}^{ir}$, then the strain-stress relation for deformation under the excavation action reads:

$$\boldsymbol{\sigma}(\underline{x}) = \boldsymbol{\sigma}^0(\underline{x}) + C\left(\underline{x}, \boldsymbol{\sigma}(\underline{x})\right) : \left[\boldsymbol{\varepsilon}(\underline{x}) - \left(\boldsymbol{\varepsilon}^{ir}(\underline{x}) - \boldsymbol{\varepsilon}^{0ir}(\underline{x})\right)\right] \tag{1.3}$$

where $\boldsymbol{\varepsilon}^{0ir}$ represents the irreversible strain at the initial state. It can be checked that if nothing is changed, then $\boldsymbol{\varepsilon}^{ir}$ remains equal to $\boldsymbol{\varepsilon}^{0ir}$, and $\boldsymbol{\varepsilon}= 0$ (non displacement because the state 0 is the origin of displacement field), and then we find well $\boldsymbol{\sigma} = \boldsymbol{\sigma}^0$.
In some problems one can suppose initial stress field that is not related to external forces, *i.e.*, can not be related to a volumetric and surface tractions that verify (1.1) and (1.2) (for instance, the internal stresses in heterogeneous materials when some thermal expansion is at the origin of the stress field). But in the study of underground openings, $\boldsymbol{\sigma}^0$ is generally related to the weight forces. In the following we suppose that $\underline{\gamma}^0$ and $\underline{T}^{0d}$ exist for $\boldsymbol{\sigma}^0$. The stress field $\boldsymbol{\sigma}$ satisfies:

$$\boldsymbol{\sigma}(\underline{x}) = \boldsymbol{\sigma}^0(\underline{x}) + C\left(\underline{x}, \boldsymbol{\sigma}(\underline{x})\right) : \left[\boldsymbol{\varepsilon}(\underline{x}) - \left(\boldsymbol{\varepsilon}^{ir}(\underline{x}) - \boldsymbol{\varepsilon}^{0ir}(\underline{x})\right)\right]$$
$$div\boldsymbol{\sigma}(\underline{x}) + \underline{\gamma} = 0,$$
$$\forall \underline{x} \in \partial_f \Omega;\ \boldsymbol{\sigma}(\underline{x}).\underline{n} = \underline{T}^d$$
$$\forall \underline{x} \in \partial_u \Omega;\ \underline{u} = \underline{u}^d$$

The numerical code takes $\underline{u}$ for the principal unknown variable. Then, this variable, or its derivative $\boldsymbol{\varepsilon}$ is deuced from the three following equations:

$$div\left(C\left(\underline{x}, \boldsymbol{\sigma}(\underline{x})\right) : \boldsymbol{\varepsilon}(\underline{x})\right) + \left(\underline{\gamma} - \underline{\gamma}^0\right) = 0,$$
$$\forall \underline{x} \in \partial_f \Omega;\ \left(C\left(\underline{x}, \boldsymbol{\sigma}(\underline{x})\right) : \boldsymbol{\varepsilon}(\underline{x})\right).\underline{n} = \underline{T}^d - \underline{T}^{d0}$$
$$\forall \underline{x} \in \partial_u \Omega;\ \underline{u} = \underline{u}^d$$

Generally, the numerical iterative method consists in supposing $\boldsymbol{\varepsilon}^{-ir}$ known, and solving the equation () and then deducing successively $\boldsymbol{\varepsilon}$, $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}^{-ir}$ respectively from (), () and irreversible deformation law, and then, iterating if necessary.
So we see that, in the modified problem, the volumetric forces have to be replaced by their difference $\underline{\gamma}$-$\underline{\gamma}^0$ and $\underline{T}^{d}$-$\underline{T}^{d0}$.
The importance of $\boldsymbol{\varepsilon}^{0ir}$ resides in its probable effect in the constitutive law. For perfect plasticity, for instance, the first equation in () can be replaced by:

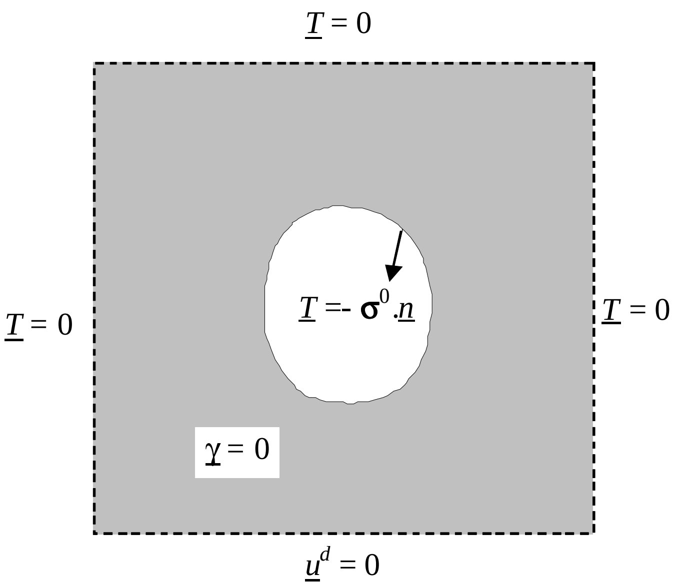
$$\boldsymbol{\sigma}(\underline{x}) = \boldsymbol{\sigma}^0(\underline{x}) + C\left(\underline{x}, \boldsymbol{\sigma}(\underline{x})\right) : \left[\boldsymbol{\varepsilon}(\underline{x}) - \boldsymbol{\varepsilon}^{ir}(\underline{x})\right]$$

And this supposes that the state ‘0’ is taken as the origin of irreversible strain. But, if we have a hardening material with hardening parameters depending on the total plastic strain, then the knowledge of $\boldsymbol{\varepsilon}^{0ir}$ is necessary to b able to determine the evolution of these parameters.
Suppose now that the aim is to calculate the displacement field due to excavation of an underground opening. In this case the original state of equilibrium under the stress $\boldsymbol{\sigma}^0$ can be taken for the origin of displacements and also irreversible deformations. The field $\boldsymbol{\sigma}^0(\underline{x})$ can be heterogeneous. In this case, in a first step a calculation is made with a formation without excavation. The value find for $\boldsymbol{\sigma}^0$ in this problem is for the reference in the following step.

If the value of $\boldsymbol{\sigma}^0$ can be known analytically, for instance as a function of depth, its value can be introduced in the calculation without much difficulty. But in heterogeneous or fractured formations, the local value of $\boldsymbol{\sigma}_0$ has to be deduced from a numerical modelling unless admitting some approximations. However for numerical calculation rise other difficulties. In Finite Elements, digging the opening is modelled by removing some elements in the mesh and then the remaining elements and nodes numbers are changed. The correspondence must be made between the nodes and elements of the initial and final mesh to transfer the $\boldsymbol{\sigma}_0$ from one to the other. Also, note that the calculation scheme is generally :

1) Determine $\underline{u}$ from Ku = F
2) Detrmine $\boldsymbol{\varepsilon}$ from $\underline{u}$ and then $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}^{0ir}$, from $\boldsymbol{\sigma}(\underline{x}) = \boldsymbol{\sigma}^0(\underline{x}) + C\left(\underline{x}, \boldsymbol{\sigma}(\underline{x})\right) : \left[\boldsymbol{\varepsilon}(\underline{x}) - \boldsymbol{\varepsilon}^{ir}(\underline{x})\right]$ with irreversible deformation laws (plastic or viscous deformations etc.)

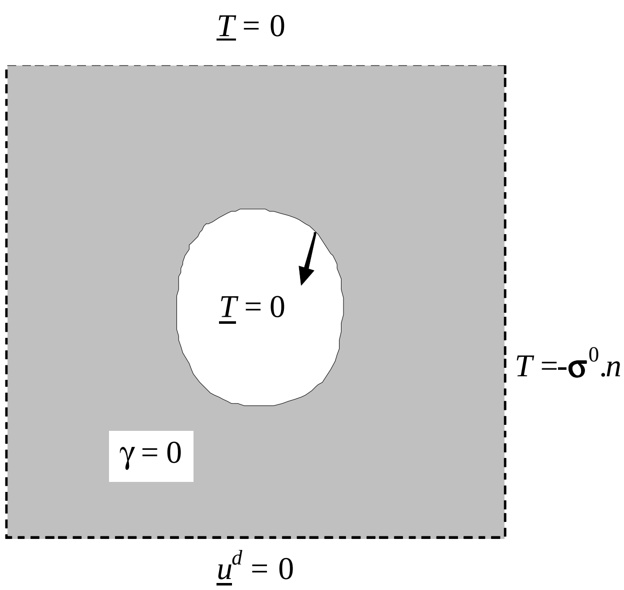
This relation supposes that $\boldsymbol{\sigma}^0(\underline{x})$ is that fixed in the first step and the numerical code can integrate it.



Sometimes the numerical code does not allow, or the user does not know how introducing an initial stress $\boldsymbol{\sigma}^0(\underline{x})$ in the computation. Then, supposing that the computation is based on the relation:

$$\boldsymbol{\sigma}(\underline{x}) = C\left(\underline{x}, \boldsymbol{\sigma}(\underline{x})\right) : \left[\boldsymbol{\varepsilon}(\underline{x}) - \boldsymbol{\varepsilon}^{ir}(\underline{x})\right],$$

the user tries generally to introduce indirectly the initial stress $\boldsymbol{\sigma}_0$ by imposing it on the boundary of the domain. The forces applied in the model are the following:



It must be noted that in this case, the displacement and strain fields include the contribution of $\boldsymbol{\sigma}_0$ and are not the same that measured during the excavation. The stress filed would be exact if there is non-linearity in the mechanical behaviour. For elastoplastic behaviour for instance, the loading path affects the final result, and therefore, imposing first the stress $\boldsymbol{\sigma}_0$ on the formation and excavating then (by imposing $-\boldsymbol{\sigma}_0.\underline{n}$ on the boundary of the excavation) does not lead to the same state of deformation than imposing (as in the figure …) the tractions -$\boldsymbol{\sigma}_0.\underline{n}$ on the boundary of the formation with and existing excavation.

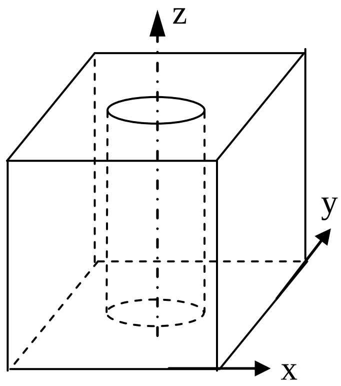
Also, the stress $\boldsymbol{\sigma}_0$ from which are derived the tractions on the boundary depends on the depth. Sometimes, it is replaced by a constant stress, and this approximation is acceptable if the excavation is enough deep so that the variation of $\boldsymbol{\sigma}_0$ between its top and bottom can be neglected relative to the mean value of $\boldsymbol{\sigma}_0$. It is clear that if this approximation can be made for horizontal borehole and galleries, it can be not accurate for some cases of underground storage caverns which have an important elongation in vertical direction (height).

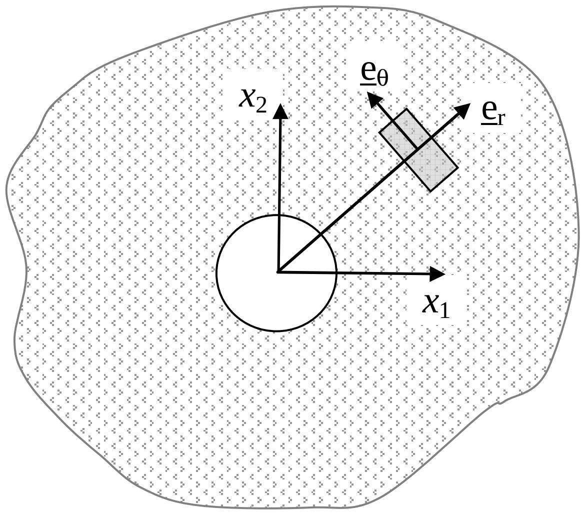
However this method can be used without problem to determine the stresses in an elastic formation containing a deep opening. In the following the stresses and displacements around cylindrical or spherical opening will be presented in elastic formations. But the approximations included in the results must be explained here.

As emphasized here above the stresses deduced from the tractions presented in the figure are exact if the formation is elastic. But this supposes also that the tractions $\underline{T}$ on the boundary derive from the exact stress $\boldsymbol{\sigma}_0$ that, as mentioned here above varied with deep. The classical results that will be presented in following sections assume a constant $\boldsymbol{\sigma}_0$. This is an approximation that can be accepted when the excavation is very deep and then the variation of $\boldsymbol{\sigma}_0$ between its top and bottom can be neglected relative to the mean value of $\boldsymbol{\sigma}_0$. So on the boundary of the domain a constant $\boldsymbol{\sigma}_0$ corresponding to the value of $\boldsymbol{\sigma}_0$ in the centre of excavation will be prescribed, what we denote by $\boldsymbol{\sigma}_0(M)$.
The second approximation consists in replacing this finite domain containing a hole with an infinite body on the boundary of with (at infinity) the constant stress $\boldsymbol{\sigma}_0(M)$. this assumption seems to be in contradiction with the first one since for a point distant from the centre of the cavity (in vertical direction) the stress can no more be taken equal to $\boldsymbol{\sigma}_0(M)$. This simplification can however be made with the counterpart that the resulting stress solution is correct only in the vicinity of the excavation.
In conclusion the following results suppose first that the opening is deep, $R << H$ where R represents the size (radius) of the opening and H its depth, and then the stresses are considered only in the vicinity of the excavation.
From this stresses the displacement field du to excavation can be deduced.

1.3 Elastic displacement and stress fields around tunnels and boreholes

We consider an infinite cylindrical hole in an infinite uniform, isotropic and linear elastic medium. (Figure). The borehole can be vertical (Figure a) or horizontal (Figure b).





The strain and stress fields have the following expressions:

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{rr} & \varepsilon_{r\theta} & \varepsilon_{rz} \\ & \varepsilon_{\theta\theta} & \varepsilon_{\theta z} \\ & & \varepsilon_{zz} \end{bmatrix} \quad \boldsymbol{\sigma} = \begin{bmatrix} \sigma_{rr} & \sigma_{r\theta} & \sigma_{rz} \\ & \sigma_{\theta\theta} & \sigma_{\theta z} \\ & & \sigma_{zz} \end{bmatrix}$$

The fundamental assumption in the modelling here is that the displacement field is radial: $\underline{u}(\underline{x}) = u(\underline{x})\,\underline{e}_r$:

$$\underline{u} = (u_r , u_\theta , u_z) = (u_r , 0, 0)$$

In addition, the radial displacement u, is function only of r and θ. (and also the $\varepsilon_{r\theta} = 0$). The result is that the components of strain and stress tensors reduce to:

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{rr} & 0 & 0 \\ 0 & \varepsilon_{\theta\theta} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{bmatrix} \quad \boldsymbol{\sigma} = \begin{bmatrix} \sigma_{rr} & 0 & 0 \\ 0 & \sigma_{\theta\theta} & 0 \\ 0 & 0 & \sigma_{zz} \end{bmatrix}$$

The relation between **ε** and **σ** is given by linear and isotropic elasticity:

$\sigma_{rr} = \lambda\, \varepsilon_{vol} + 2G\, \varepsilon_{rr}$, $\quad \sigma_{\theta\theta} = \lambda\, \varepsilon_{vol} + 2G\, \varepsilon_{\theta\theta}$, $\quad \sigma_{zz} = \lambda\, \varepsilon_{vol} + 2G\, \varepsilon_{zz}$

Where : $\varepsilon_{vol} = \varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{zz}$

1.3.1 Equilibrium conditions

The equilibrium equations that are $\partial_i \sigma_{ij} + f_j = 0$
and more explicitly :

$$\partial_x\sigma_{xx} + \partial_y\sigma_{yx} + \partial_z\sigma_{zx} + f_x = 0$$
$$\partial_x\sigma_{xy} + \partial_y\sigma_{yy} + \partial_z\sigma_{zy} + f_y = 0$$
$$\partial_x\sigma_{xz} + \partial_y\sigma_{yz} + \partial_z\sigma_{zz} + f_z = 0$$

in cylindrical coordinates become :

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta} + \frac{\partial\sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \mathrm{f}_r = 0$$
$$\frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta} + \frac{\partial\sigma_{r\theta}}{\partial r} + \frac{\partial\sigma_{\theta z}}{\partial z} + \frac{2\sigma_{r\theta}}{r} + \mathrm{f}_\theta = 0$$
$$\frac{\partial\sigma_{zz}}{\partial z} + \frac{\partial\sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\theta z}}{\partial\theta} + \frac{\sigma_{rz}}{r} + \mathrm{f}_z = 0$$

For the case of stress fields around the borehole, these equations reduce to ($\sigma_{z\theta}=\sigma_{rz}=0$, $\underline{f} = 0$, independent of z):

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0$$
$$\frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta} + \frac{\partial\sigma_{r\theta}}{\partial r} + \frac{2\sigma_{r\theta}}{r} = 0$$

1.3.2 The boundary conditions

There are two sets of boundary conditions:

-At the borehole wall, r = R: $\sigma_{rr}=\sigma_{r\theta}=\sigma_{rz}= 0$ for every θ
- for the limit $r \to \infty$, $\boldsymbol{\sigma}(r,\theta,z) \to \boldsymbol{\sigma}_0$
$\boldsymbol{\sigma}_0$ is the constant stress tensor representing the underground stresses before digging the borehole, or also far from the borehole.
In the Cartesian coordinate system:

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} & 0 & 0 \\ 0 & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_{zz} \end{bmatrix}$$

But, note that, the relation between the stress components in Cartesian and cylindrical coordinates is the following:

$$\sigma_{rr} = \underline{e}_r.\boldsymbol{\sigma}.\underline{e}_r = \begin{pmatrix}\cos\theta & \sin\theta & 0\end{pmatrix} \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \begin{pmatrix}\cos\theta \\ \sin\theta \\ 0\end{pmatrix} = \sigma_{xx}\cos^2\theta + 2\sigma_{xy}\sin\theta\cos\theta + \sigma_{yy}\sin^2\theta$$

$$\sigma_{\theta\theta} = \underline{e}_\theta.\boldsymbol{\sigma}.\underline{e}_\theta = \begin{pmatrix}-\sin\theta & \cos\theta & 0\end{pmatrix}\begin{bmatrix}\sigma_{xx} & \sigma_{xy} & \sigma_{xz}\\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz}\\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz}\end{bmatrix}\begin{pmatrix}-\sin\theta\\ \cos\theta\\ 0\end{pmatrix} = \sigma_{xx}\sin^2\theta - 2\sigma_{xy}\sin\theta\cos\theta + \sigma_{yy}\cos^2\theta$$

$$\sigma_{r\theta} = \underline{e}_r.\boldsymbol{\sigma}.\underline{e}_\theta = \begin{pmatrix}\cos\theta & \sin\theta & 0\end{pmatrix}\begin{bmatrix}\sigma_{xx} & \sigma_{xy} & \sigma_{xz}\\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz}\\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz}\end{bmatrix}\begin{pmatrix}-\sin\theta\\ \cos\theta\\ 0\end{pmatrix} = (\sigma_{yy} - \sigma_{xx})\sin\theta\cos\theta + \sigma_{xy}(\cos^2\theta - \sin^2\theta)$$

Then for ($\sigma_{xx} = \sigma_1$, $\sigma_{yy} = \sigma_2$, $\sigma_{xy} = 0$) at infinity, we find the following boundary conditions:

Limit for $r \to \infty$:

$$\sigma_{rr} \to \sigma_1\cos^2\theta + \sigma_2\sin^2\theta$$
$$\sigma_{\theta\theta} \to \sigma_1\sin^2\theta + \sigma_2\cos^2\theta$$
$$\sigma_{r\theta} \to \frac{\sigma_2 - \sigma_1}{2}\sin 2\theta$$

1.3.3 Resolution of the equations

The method of resolution consists in expressing the equations in terms of the displacement $\underline{u}$. The equilibrium equation expressed in terms of u gives:

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0 \to \frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} + \frac{b}{\lambda + 2G}\frac{dp}{dr} = 0$$

We consider first the case of a hollow cylinder without the effect of pressure, or with constant pressure. The solution of u can be written as:

$$u = C_1 r + \frac{C_2}{r}$$

Where C_1 and C_2 are two constants which have to be determined from the boundary conditions. The expression of stresses is found to be:

$$\sigma_{rr} = A - \frac{B}{r^2}, \qquad \sigma_{\theta\theta} = A + \frac{B}{r^2}$$

Where A and B are constant. The boundary condition ($\sigma_{rr} = \sigma_{in}$ at R_i and $\sigma_{rr} = \sigma_{ex}$ at R_e) lead to:

$$\sigma_{rr} = \sigma_i + \frac{R_i^2(\sigma_e - \sigma_i)}{R_e^2 - R_i^2}\left(1 - \frac{R_i^2}{r^2}\right), \qquad \sigma_{\theta\theta} = \sigma_i + \frac{R_i^2(\sigma_e - \sigma_i)}{R_e^2 - R_i^2}\left(1 + \frac{R_i^2}{r^2}\right),$$

When $R_e/R_i \to \infty$, the solution be come (noting $R = R_i$):

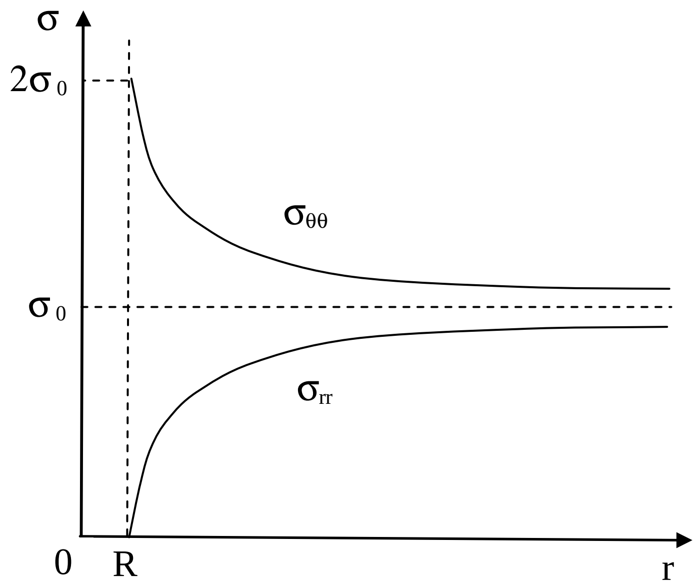
$$\sigma_{rr} = \sigma_i + (\sigma_e - \sigma_i)\left(1 - \frac{R^2}{r^2}\right), \qquad \sigma_{\theta\theta} = \sigma_i + (\sigma_e - \sigma_i)\left(1 + \frac{R^2}{r^2}\right)$$

The vertical stress around the borehole, σ_{zz} is initially equal to σ_v. The variation of the stress correspond to zero vertical strain: $\delta\varepsilon_{zz} = (\delta\sigma_{zz} - \nu\delta\sigma_{rr} - \nu\delta\sigma_{\theta\theta}) = 0$. So:
$\delta\sigma_{zz} = \nu(\delta\sigma_{rr} + \delta\sigma_{\theta\theta}) = \nu(\sigma_{rr} + \sigma_{\theta\theta} - 2\sigma_h) = \nu(2\sigma_i + 2(\sigma_e - \sigma_i) - 2\sigma_h) = 0$ since $\sigma_e = \sigma_h$.
So $\sigma_{zz} = \sigma_v$.

1.3.4 Simple case of borehole stresses

If the pressure in the borehole is zero and the infinite stress is σ_0, then the stress is given by:

$$\sigma_{rr} = \sigma_0\left(1-\frac{R^2}{r^2}\right), \qquad \sigma_{\theta\theta} = \sigma_0\left(1+\frac{R^2}{r^2}\right), \qquad \sigma_{r\theta} = 0$$



Stress around a borehole for isotropic stress σ_0 at infinity

at the borehole wall:

$$\sigma_{rr} = 0 \qquad \sigma_{\theta\theta} = 2\sigma_0 \qquad \sigma_{zz} = \sigma_v$$

If the underground stresses are isotropic $\sigma_0 = \sigma_h = \sigma_H = \sigma_v$
Then the three principal stresses are: $\sigma_{rr} = 0 \qquad \sigma_{\theta\theta} = 2\sigma_v \qquad \sigma_{zz} = \sigma_v$
If the compression strength is C_0 then the failure occurs when $2\sigma_v = C_0$.
For instance, if C_0 = 10 MPa (chalk), the limit depth of the borehole allowed by the stability condition would be given by σ_v = 5 MPa, so H = 200m if the average density is γ =2.5
But, of course, this analyses does note take account of the effect of the borehole mud. This question will be studied in more details in the following.

1.3.5 Borehole at pressure p

The case of the borehole at pressure p can be deduced from the above formulae with $\sigma_i = p_w$.
It can be obtained also by superposition principle of an isotropic stress:

Consider, the stress field corresponding to an internal pressure p in the borehole and zero stress at infinity. It is obtained by *superposition* of a constant and isotropic stress field p:

$$\boldsymbol{\sigma} = \begin{bmatrix} p & \\ & p \end{bmatrix}$$

and the above stress filed for σ_0 = -p. The solution is:

$$\sigma_{rr} = -p + p\left(1 - \frac{R^2}{r^2}\right), \qquad \sigma_{\theta\theta} = -p + p\left(1 + \frac{R^2}{r^2}\right), \qquad \sigma_{r\theta} = 0$$

$$\sigma_{rr} = -p\frac{R^2}{r^2}, \qquad \sigma_{\theta\theta} = p\frac{R^2}{r^2}, \qquad \sigma_{r\theta} = 0$$

1.3.6 The stress field solution for anisotropic far-field stress

A solution to the above equations is obtained by solving the equations on the displacement field. The solution of stress field is is the following. It must easily be checked that it satisfies the equilibrium and boundary conditions:

$$\sigma_{rr} = \sigma_1 \cos^2\theta + \sigma_2 \sin^2\theta - \frac{\sigma_1 + \sigma_2}{2}\frac{R^2}{r^2} + \frac{\sigma_2 - \sigma_1}{2}\left(4 - \frac{3R^2}{r^2}\right)\frac{R^2}{r^2}\cos 2\theta$$

$$\sigma_{\theta\theta} = \sigma_1 \sin^2\theta + \sigma_2 \cos^2\theta + \frac{\sigma_1 + \sigma_2}{2}\frac{R^2}{r^2} + \frac{3(\sigma_2 - \sigma_1)}{2}\frac{R^4}{r^4}\cos 2\theta$$

$$\sigma_{r\theta} = \frac{\sigma_2 - \sigma_1}{2}\left[1 + \left(2 - \frac{3R^2}{r^2}\right)\frac{R^2}{r^2}\right]\sin 2\theta$$

Now, we can superpose this stress field to that corresponding to (σ_1, σ_2) to find the stress field in a borehole with internal pressure p and underground initial stresses (σ_1, σ_2). We find:

$$\sigma_{rr} = \sigma_1 \cos^2\theta + \sigma_2 \sin^2\theta - \frac{\sigma_1 + \sigma_2}{2}\frac{R^2}{r^2} + \frac{\sigma_2 - \sigma_1}{2}\left(4 - \frac{3R^2}{r^2}\right)\frac{R^2}{r^2}\cos 2\theta - p\frac{R^2}{r^2}$$

$$\sigma_{\theta\theta} = \sigma_1 \sin^2\theta + \sigma_2 \cos^2\theta + \frac{\sigma_1 + \sigma_2}{2}\frac{R^2}{r^2} + \frac{3(\sigma_2 - \sigma_1)}{2}\frac{R^4}{r^4}\cos 2\theta + p\frac{R^2}{r^2}$$

$$\sigma_{r\theta} = \frac{\sigma_2 - \sigma_1}{2}\left[1 + \left(2 - \frac{3R^2}{r^2}\right)\frac{R^2}{r^2}\right]\sin 2\theta$$

1.3.7 Stress state at the borehole wall

The risk of failure is maximal at the borehole wall. On the wall of the borehole the stress components are:

$$\sigma_{rr} = -p$$
$$\sigma_{\theta\theta} = (3\sigma_2 - \sigma_1)\cos^2\theta + (3\sigma_1 - \sigma_2)\sin^2\theta + p$$
$$\sigma_{r\theta} = 0$$

As previously the stress in the axial direction σ_{zz}, is obtained by assuming plane strain deformation during the drillout process of the borehole. The initial stress state is (σ_v, σ_1, σ_2) and the finale (σ_{zz}, σ_{rr}, $\sigma_{\theta\theta}$), and since $\delta\sigma_{zz} = \nu(\sigma_{rr} + \sigma_{\theta\theta} - \sigma_1 - \sigma_2) = 2\nu(\sigma_2 - \sigma_1)\cos 2\theta$

So :

$$\sigma_{zz} = \sigma_v + 2\nu(\sigma_2 - \sigma_1)\cos 2\theta$$

The extreme values of $\sigma_{\theta\theta}$ are:
$\theta = 0$: $\sigma_{\theta\theta} = 3\sigma_2 - \sigma_1 - p$ $\sigma_{zz} = \sigma_v + 2\nu(\sigma_2 - \sigma_1)$
$\theta = \pi/2$: $\sigma_{\theta\theta} = 3\sigma_1 - \sigma_2 - p$ $\sigma_{zz} = \sigma_v - 2\nu(\sigma_2 - \sigma_1)$

If the underground stress is isotropic, $\sigma_1 = \sigma_2 = \sigma_0$ then $\sigma_{\theta\theta}$ is independent of θ and equal to:
$\sigma_{rr} = p$ $\sigma_{\theta\theta} = 2\sigma_0 - p$, $\sigma_{zz} = \sigma_v$ for every θ

1.3.8 Displacement field

The displacement is counted from the state of $\boldsymbol{\sigma}_0$ stress. The stress $\boldsymbol{\sigma}_0$ induces a linear displacement field that is taken to zero at the borehole centre.

This displacement is given by:

$$u_r = \frac{1-\nu^2}{E}\left[\frac{\sigma_1+\sigma_2}{2}\left(r+\frac{R^2}{r}\right)+\frac{\sigma_2-\sigma_1}{2}\left(r+\frac{4R^2}{r}-\frac{R^4}{r^3}\right)\cos 2\theta\right]$$
$$-\frac{\nu(1+\nu)}{E}\left[\frac{\sigma_1+\sigma_2}{2}\left(r-\frac{R^2}{r}\right)-\frac{\sigma_2-\sigma_1}{2}\left(r-\frac{R^4}{r^3}\right)\cos 2\theta\right]+p\,????\frac{R^2}{r}$$
$$u_\theta = -\frac{\sigma_2-\sigma_1}{2}\frac{(1+\nu)}{E}\left\{\frac{2R^2}{r}+(1-2\nu)\left(r+\frac{R^4}{r^3}\right)\right\}\sin 2\theta$$

To this displacement must be added

$\underline{u} = \boldsymbol{\varepsilon}_0.\underline{x}$

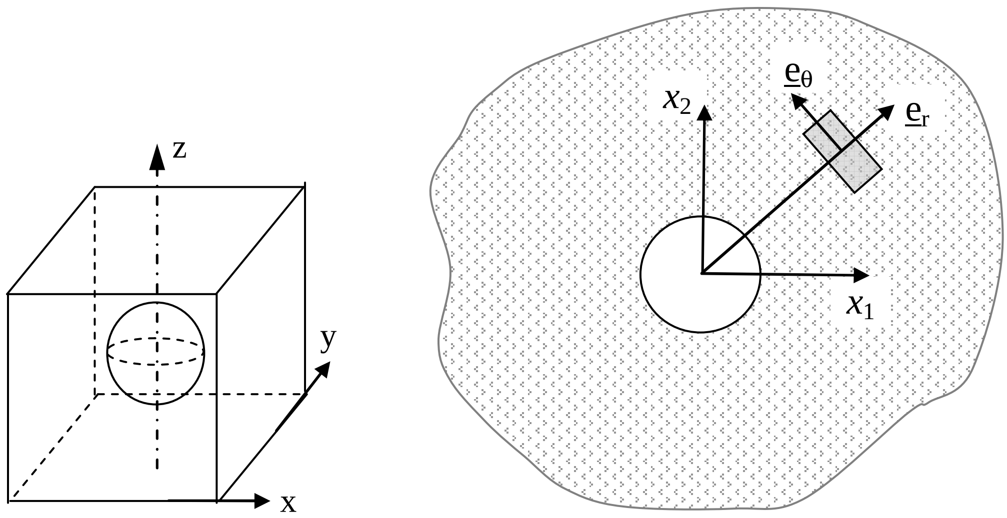
We have $\boldsymbol{\varepsilon}_0 = \begin{bmatrix} \varepsilon_1 & \\ & \varepsilon_2 \end{bmatrix}$ with $\varepsilon_1 = \dfrac{\sigma_1 - \nu\sigma_2}{E}$, $\varepsilon_2 = \dfrac{\sigma_2 - \nu\sigma_1}{E}$ and then, in cylindrical coordinates:

$$u_r^0 = \frac{(\sigma_1+\sigma_2)(1-\nu)}{2E}r - \frac{(\sigma_2-\sigma_1)(1+\nu)}{2E}r\cos 2\theta$$
$$u_\theta^0 = \frac{1+\nu}{2E}(\sigma_2-\sigma_1)\,r\sin 2\theta$$

1.4 Elastic displacement and stress fields around a deep underground spherical cavity

To determine this state of stress we have to consider the spherical geometry of the cavity and to write the above equations in a spherical system of coordinates.

We consider a spherical hole in an infinite uniform, isotropic and linear elastic medium. (Figure).



The strain and stress fields have the following expressions:

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{rr} & \varepsilon_{r\theta} & \varepsilon_{r\varphi} \\ & \varepsilon_{\theta\theta} & \varepsilon_{\theta\varphi} \\ & & \varepsilon_{\varphi\varphi} \end{bmatrix} \qquad \boldsymbol{\sigma} = \begin{bmatrix} \sigma_{rr} & \sigma_{r\theta} & \sigma_{r\varphi} \\ & \sigma_{\theta\theta} & \sigma_{\theta\varphi} \\ & & \sigma_{\varphi\varphi} \end{bmatrix}$$

The assumption of spherical symmetry implies that the displacement field is radial: $\underline{u}(\underline{x}) = u(\underline{x})\, \underline{e}_r$:

$$\underline{u} = (u_r\, , u_\theta\, , u_z) = (u_r\, , 0, 0)$$

In addition, the radial displacement u, is function only of r. The result is that the off-diagonal components of strain and stress tensors vanish, and moreover $\varepsilon_{\theta\theta}=\varepsilon_{\varphi\varphi}$ and $\sigma_{\theta\theta} = \sigma_{\varphi\varphi}$:

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{rr} & 0 & 0 \\ 0 & \varepsilon_{\theta\theta} & 0 \\ 0 & 0 & \varepsilon_{\theta\theta} \end{bmatrix} \qquad \boldsymbol{\sigma} = \begin{bmatrix} \sigma_{rr} & 0 & 0 \\ 0 & \sigma_{\theta\theta} & 0 \\ 0 & 0 & \sigma_{\theta\theta} \end{bmatrix}$$

Moreover:

$$\varepsilon_{rr} = \frac{\partial u}{\partial r}, \qquad \varepsilon_{\theta\theta} = \varepsilon_{\varphi\varphi} = \frac{u}{r}$$

The relation between $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$ is given by linear and isotropic elasticity:

$$\sigma_{rr} = \lambda\, \varepsilon_{vol} + 2G\, \varepsilon_{rr}\, , \qquad \sigma_{\theta\theta} = \lambda\, \varepsilon_{vol} + 2G\, \varepsilon_{\theta\theta}\, , \qquad \sigma_{zz} = \lambda\, \varepsilon_{vol} + 2G\, \varepsilon_{zz}$$

where:

$$\varepsilon_{vol} = \varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{zz}$$

Equilibrium conditions

The equilibrium equations that are $\quad \partial_i\sigma_{ij} + f_j = 0$
and more explicitly :

$$\partial_x\sigma_{xx} + \partial_y\sigma_{yx} + \partial_z\sigma_{zx} + f_x = 0$$
$$\partial_x\sigma_{xy} + \partial_y\sigma_{yy} + \partial_z\sigma_{zy} + f_y = 0$$
$$\partial_x\sigma_{xz} + \partial_y\sigma_{yz} + \partial_z\sigma_{zz} + f_z = 0$$

in spherical coordinates become :

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\sigma_{r\phi}}{\partial\phi} + \frac{2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\phi\phi} + \sigma_{r\theta}\cot\theta}{r} + f_r = 0$$
$$\frac{1}{r}\frac{\partial\sigma_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\sigma_{\theta\phi}}{\partial\phi} + \frac{(\sigma_{\theta\theta} - \sigma_{\phi\phi})\cot\theta + 3\sigma_{r\theta}}{r} + f_\theta = 0$$
$$\frac{1}{r}\frac{\partial\sigma_{\phi r}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\phi\theta}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\sigma_{\phi\phi}}{\partial\phi} + \frac{3\sigma_{r\phi} + 2\sigma_{\theta\phi}\cot\theta}{r} + f_\phi = 0$$

For the case of stress fields around a spherical cavity, under spherical symmetry conditions, ($\sigma_{r\theta}=\sigma_{r\phi}=0$, $\sigma_{\theta\theta}=\sigma_{\phi\phi}$, $\underline{f} = 0$, independent of θ and ϕ):

$$\frac{\partial\sigma_{rr}}{\partial r} + 2\frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0$$

The boundary conditions

There are two sets of boundary conditions:

-At the borehole wall, r = R: $\quad \sigma_{rr}=\sigma_{r\theta}=\sigma_{r\phi} = 0$ for every θ,ϕ

- for the the limite $r \to \infty$, $\quad \boldsymbol{\sigma}(r,\theta,\phi) \to \boldsymbol{\sigma}_0 = -P_e\,\boldsymbol{\delta}$
$\boldsymbol{\sigma}_0$ is the constant stress tensor representing the underground stresses before digging the borehole, or also far from the borehole.

Resolution of the equations
The method of resolution consists in expressing the equations in terms of the displacement $\underline{u}$.
For elastic behaviour:
We take $\underline{u}(r) = u(r)\,\underline{e}_r$,
And, by replacing this expression of the strain and stress field for linear elasticity, and in the equilibrium equation, we find:

$$\frac{\partial\sigma_{rr}}{\partial r} + 2\frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0 \rightarrow (\lambda + 2\mu)\left[4\frac{d}{dr}\left(\frac{u}{r}\right) + r\frac{d^2}{dr^2}\left(\frac{u}{r}\right)\right] = 0$$

The general solution of this equation is:

$$u = \alpha r + \frac{\beta}{r^2}$$

where *a* and *b* are two constants to be determined from the boundary conditions. Then we find:

$$\varepsilon_{rr} = \alpha - \frac{2\beta}{r^3}, \qquad \varepsilon_{\theta\theta} = \varepsilon_{\varphi\varphi} = \alpha + \frac{\beta}{r^3}$$

and:

$$\sigma_{rr} = A - \frac{2B}{r^3}, \qquad \sigma_{\theta\theta} = \sigma_{\varphi\varphi} = A + \frac{B}{r^3}$$

where the constants A and B are given by:

$$A = (3\lambda+2\mu)\,\alpha \quad , \quad B = 2\mu\beta$$

Hollow Sphere

We consider first the case of a hollow sphere without the effect of pressure, or with constant pressure. In this case, the boundary conditions ($\sigma_{rr}= -p_i$ at R_i and $\sigma_{rr}= -p_{ex}$ at R_e) lead to:

$$A = \frac{p_i R_i^3 - p_e R_e^3}{R_e^3 - R_i^3}, \qquad B = \frac{1}{2}\frac{(p_i - p_e)R_i^3 R_e^3}{R_e^3 - R_i^3}$$

When $R_e/R_i \rightarrow \infty$, the solution become (noting $R = R_i$):

$$A = -p_e, \qquad B = \frac{1}{2}(p_i - p_e)R_i^3$$

and then with CM Convention (compression negative):

$$\sigma_{rr} = -p_e + \frac{(p_e - p_i)R_i^3}{r^3}, \qquad \sigma_{\theta\theta} = \sigma_{\varphi\varphi} = -p_e - \frac{(p_e - p_i)R_i^3}{2r^3}$$

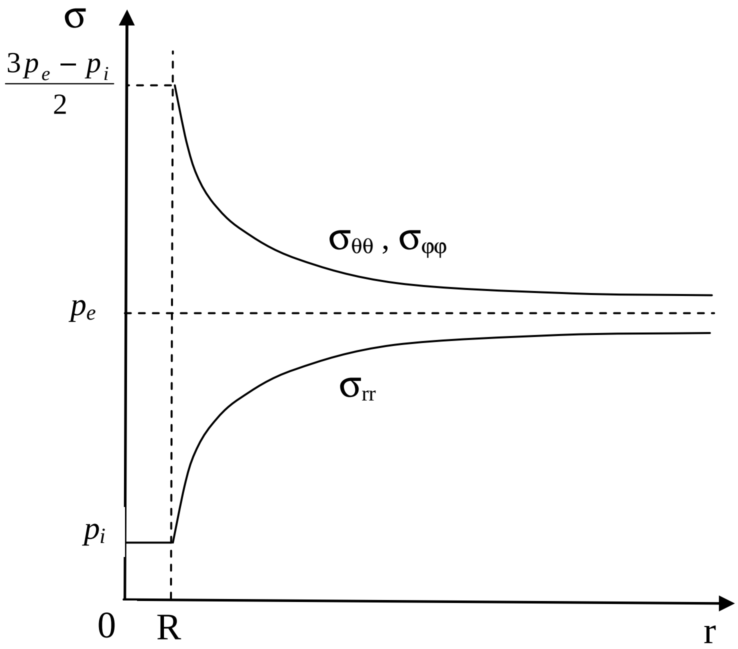
The displacement is given by (*u* positive if outward):

$$u = \frac{-p_e}{3\lambda + 2\mu} r + \frac{(p_i - p_e)R_i^3}{4\mu r^2}$$

If the compressive stresses are denoted positively (RM Convention):

$$\sigma_{rr} = p_e - \frac{(p_e - p_i)R_i^3}{r^3}, \qquad \sigma_{\theta\theta} = \sigma_{\varphi\varphi} = p_e + \frac{(p_e - p_i)R_i^3}{2r^3}$$

The variation of σ_{rr} and $\sigma_{\theta\theta}$ are represented in the following figure.



Stress around a cavity for isotropic stress $\sigma_0 = p_e$ at infinity

If the initial stress state of the ground before cavity excavation, corresponding to a uniform ground stress $-p_e\boldsymbol{\delta}$, is taken as the origin of displacements, then only the displacement reduces to:

$$u = \frac{(p_i - p_e)R_i^3}{4\mu r^2}$$

And the convergence (displacement at the cavity wall counted positively if inward) will be:

$$\frac{u_i}{R_i} = \frac{1+\nu}{2E}(p_e - p_i)$$

Appendix

Elastic stress and strain fields solutions around boreholes

A set of elementary homogeneous displacement fields satisfying equilibrium equations with zero forces are presented below. In cylindrical coordinates, in plane stress or plane strain conditions, the strain field is given by:

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad \varepsilon_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}$$

$$\varepsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)$$

The Hooke's law becomes:

Plane Strain

$$\sigma_{rr} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{rr} + \nu\varepsilon_{\theta\theta}\right]$$

$$\sigma_{\theta\theta} = \frac{E}{(1+\nu)(1-2\nu)}\left[\nu\varepsilon_{rr} + (1-\nu)\varepsilon_{\theta\theta}\right]$$

$$\sigma_{r\theta} = \frac{E}{1+\nu}\varepsilon_{r\theta}, \quad \sigma_{zz} = \nu(\sigma_{rr} + \sigma_{\theta\theta})$$

Plane Stress

$$\sigma_{rr} = \frac{E}{1-\nu^2}(\varepsilon_{rr} + \nu\varepsilon_{\theta\theta})$$

$$\sigma_{\theta\theta} = \frac{E}{1-\nu^2}(\nu\varepsilon_{rr} + \varepsilon_{\theta\theta})$$

$$\sigma_{r\theta} = \frac{E}{1+\nu}\varepsilon_{r\theta}, \quad \sigma_{zz} = 0$$

The equilibrium equations reduce in:

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0$$

$$\frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{2\sigma_{r\theta}}{r} = 0$$

Let the following displacement fields with associated stresses in *plane strain* conditions be considered:

0) Linear displacement field, uniform strain and stress:

$$u_r^0 = \frac{1+\nu}{E}\left[\frac{\sigma_1+\sigma_2}{2}(1-2\nu)r - \frac{\sigma_2-\sigma_1}{2}r\cos 2\theta\right]$$

$$u_\theta^0 = \frac{(1+\nu)}{E}\frac{\sigma_2-\sigma_1}{2}r\sin 2\theta$$

$$\sigma_{rr}^0 = \frac{\sigma_1+\sigma_2}{2} - \frac{\sigma_2-\sigma_1}{2}\cos 2\theta$$

$$\sigma_{\theta\theta}^0 = \frac{\sigma_1+\sigma_2}{2} + \frac{\sigma_2-\sigma_1}{2}\cos 2\theta$$

$$\sigma_{r\theta}^0 = \frac{\sigma_2-\sigma_1}{2}\sin 2\theta$$

1) Displacement in r^{-1}, due to a "pressure" in the borehole

$$u_r^1 = a_1\frac{R^2}{r} \qquad \sigma_{rr}^1 = -a_1\frac{E}{1+\nu}\frac{R^2}{r^2}$$

$$u_\theta^1 = 0 \qquad \sigma_{\theta\theta}^1 = a_1\frac{E}{1+\nu}\frac{R^2}{r^2}, \quad \sigma_{r\theta}^1 = 0$$

2) Pur shear displacement field homogeneous in r^{-3}:

$$u_r^2 = a_2 \frac{R^4}{r^3}\cos 2\theta \qquad \sigma_{rr}^2 = -a_2 \frac{3E}{1+\nu}\frac{R^4}{r^4}\cos 2\theta$$

$$u_\theta^2 = a_2 \frac{R^4}{r^3}\sin 2\theta \qquad \sigma_{\theta\theta}^2 = a_2 \frac{3E}{1+\nu}\frac{R^4}{r^4}\cos 2\theta$$

$$\sigma_{r\theta}^2 = -a_2 \frac{3E}{1+\nu}\frac{R^4}{r^4}\sin 2\theta$$

3) Auxiliary displacement field homogeneous in r^{-1}:

$$u_r^3 = 2a_3(1-\nu)\frac{R^2}{r}\cos 2\theta \qquad \sigma_{rr}^3 = -a_3 \frac{2E}{1+\nu}\frac{R^2}{r^2}\cos 2\theta$$

$$\sigma_{\theta\theta}^3 = 0$$

$$u_\theta^3 = -a_3(1-2\nu)\frac{R^2}{r}\sin 2\theta \qquad \sigma_{r\theta}^3 = -a_3 \frac{E}{1+\nu}\frac{R^2}{r^2}\sin 2\theta$$

It is easy to check that all these elementary homogeneous fileds satisfy individually the equilibrium conditions when the stresses are computed by the *plane strain* Hooke's law.

The displacement $\underline{u}^0$ corresponds to the uniform stress filed $\boldsymbol{\sigma}^{0:}$

$$\boldsymbol{\sigma}^0 = \begin{bmatrix} \sigma_1 & \\ & \sigma_2 \end{bmatrix}$$

The stress fields $\boldsymbol{\sigma}^1$ to $\boldsymbol{\sigma}^3$ corresponding to the other elementary solutions vanish at infinity. To build the stress field around the borehole after its excavation, a combination of displacements $\boldsymbol{\sigma}^1$ to $\boldsymbol{\sigma}^3$ must be superposed $\boldsymbol{\sigma}^0$ allowing to vanish at the borehole boundary the stress $\boldsymbol{\sigma}^0.\underline{e}_r$ and add the contribution of a pressure denoted by σ_p.

Let denote by

$$\bar{\boldsymbol{\sigma}} = -\boldsymbol{\sigma}^0 + \sigma_p \boldsymbol{\delta} = \begin{bmatrix} -\sigma_1 + \sigma_p & \\ & -\sigma_2 + \sigma_p \end{bmatrix}$$

This corresponds to:

$$\bar{\sigma}_{rr} = -\frac{\sigma_1+\sigma_2}{2} + \sigma_p + \frac{\sigma_2-\sigma_1}{2}\cos 2\theta$$

$$\bar{\sigma}_{\theta\theta} = -\frac{\sigma_1+\sigma_2}{2} + \sigma_p - \frac{\sigma_2-\sigma_1}{2}\cos 2\theta$$

$$\bar{\sigma}_{r\theta} = -\frac{\sigma_2-\sigma_1}{2}\sin 2\theta$$

Only the components $\bar{\sigma}_{rr}$ and $\bar{\sigma}_{r\theta}$ have to be created with the fields $\boldsymbol{\sigma}^1$ to $\boldsymbol{\sigma}^3$. By computing the values of $\boldsymbol{\sigma}^1$ to $\boldsymbol{\sigma}^3$ at $r = R$, the coefficients of the combination are determined from the following equations that must be satisfied for every θ :

$$\bar{\sigma}_{rr} = -\frac{\sigma_1+\sigma_2}{2} + \sigma_p + \frac{\sigma_2-\sigma_1}{2}\cos 2\theta = -\frac{E}{1+\nu}\left[a_1 + (3a_2+2a_3)\cos 2\theta\right]$$
$$\bar{\sigma}_{r\theta} = -\frac{\sigma_2-\sigma_1}{2}\sin 2\theta = -\frac{E}{1+\nu}(3a_2+a_3)\sin 2\theta$$

This yields:

$$a_1 = \frac{1+\nu}{E}\left(\frac{\sigma_1+\sigma_2}{2} - \sigma_p\right)$$
$$a_2 = \frac{1+\nu}{E}\frac{\sigma_2-\sigma_1}{2}$$
$$a_3 = -\frac{2(1+\nu)}{E}\frac{\sigma_2-\sigma_1}{2}$$

With these coefficients, the displacement filed $\underline{\bar{u}} = \underline{u}^1 + \underline{u}^2 + \underline{u}^3$ associated to $\bar{\boldsymbol{\sigma}}$, and representing the displacement due to excavation and pressurization of the borehole become:

$$\bar{u}_r = \frac{1+\nu}{E}\left[\frac{\sigma_1+\sigma_2}{2}\frac{R^2}{r} - \frac{\sigma_2-\sigma_1}{2}\left(4(1-\nu)\frac{R^2}{r} - \frac{R^4}{r^3}\right)\cos 2\theta\right] - \frac{1+\nu}{E}\sigma_p\frac{R^2}{r}$$
$$\bar{u}_\theta = \frac{1+\nu}{E}\frac{\sigma_2-\sigma_1}{2}\left[2(1-2\nu)\frac{R^2}{r} + \frac{R^4}{r^3}\right]\sin 2\theta$$

And the stress filed is:

$$\sigma_{rr} = \frac{\sigma_1+\sigma_2}{2}\left(1-\frac{R^2}{r^2}\right) - \frac{\sigma_2-\sigma_1}{2}\left(1-4\frac{R^2}{r^2}+3\frac{R^4}{r^4}\right)\cos 2\theta + \sigma_p\frac{R^2}{r^2}$$
$$\sigma_{\theta\theta} = \frac{\sigma_1+\sigma_2}{2}\left(1+\frac{R^2}{r^2}\right) + \frac{\sigma_2-\sigma_1}{2}\left(1+3\frac{R^4}{r^4}\right)\cos 2\theta - \sigma_p\frac{R^2}{r^2}$$
$$\sigma_{r\theta} = \frac{\sigma_2-\sigma_1}{2}\left(1+2\frac{R^2}{r^2}-3\frac{R^4}{r^4}\right)\sin 2\theta$$

Note that the displacement associated to $\boldsymbol{\sigma}$ is $\underline{u}$ = $\underline{u} = \underline{\bar{u}} + \underline{u}^0$, but only $\underline{\bar{u}}$ represents the real displacement due to excavation and pressurization and can be measured. $\underline{u}^0$ has no physical meaning and tends to infinity when $r \to \infty$. The Hook's law reads for this case of existing initial stresses:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_0 + \mathcal{C} : \boldsymbol{\varepsilon}$$

where $\boldsymbol{\varepsilon}$ is the strain associated to $\underline{\bar{u}}$ For the general convention of continuum mechanics, (compression negative) we have $\sigma_p = -p$ where p designate the pressure (positive) in the borehole and the positive u_r points outward the borehole. For the Soil Mechanics convention (compression counted positively), we have to take $\sigma_p = p$ and then positive u_r points inward the borehole.